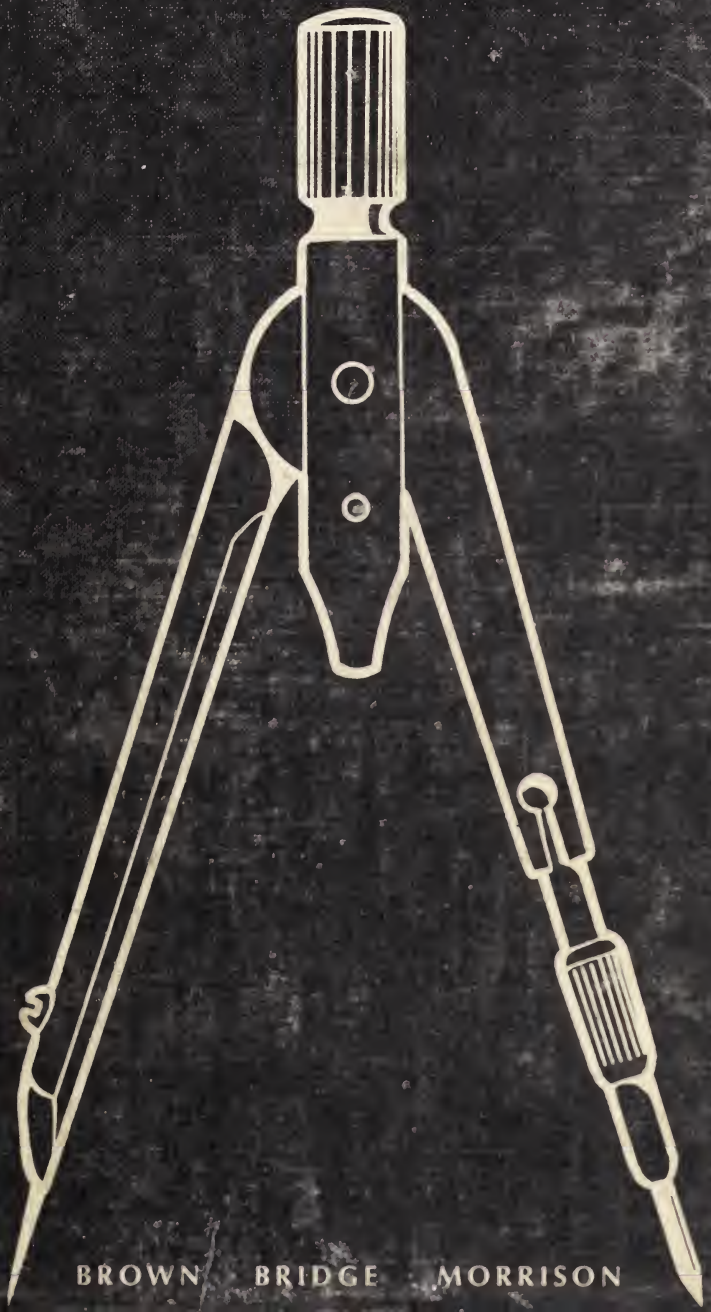
MATHEMATICS IN PRACTICE

REVISED EDITION



BROWN BRIDGE MORRISON

NOTICE

This book is being lent to you subject to the foll

1

2

Ex LIBRIS
UNIVERSITATIS
ALBERTAENSIS

3

Not

4.

STUDENT'S NAME

FILE NUMBER

DO NOT DETACH THIS SLIP. Enter your name in the space to the left, and keep the slip in the book so that you may be credited with its return.

Return the book by insured mail when your registration expires, or when your course has been completed, discontinued or cancelled.

FOR OFFICE USE ONLY

Above-named student credited with the return of:

Following not returned:

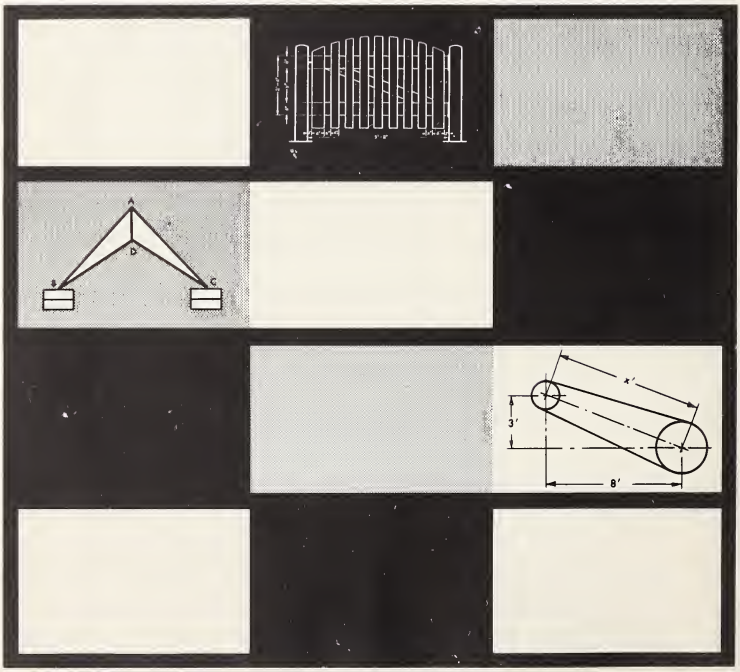
Correspondence
Department of Education
Edmonton - Alberta
PROVINCIAL ARCHIVES
OF ALBERTA
ACC. 73.225

Digitized by the Internet Archive
in 2017 with funding from
University of Alberta Libraries

https://archive.org/details/mathematicsinpra00brow

MATHEMATICS IN PRACTICE

MATHEMATICS



THE MACMILLAN COMPANY

IN PRACTICE

REVISED
EDITION

Authorized in the Province of Alberta	
Principal, Danforth Technical School, Toronto	**ARTHUR E. BROWN** B.A., B. PAED.
Technical Training Specialist, Vocational Training Branch, Department of Labour, Federal Government	**DAVID E. BRIDGE** B.A. SC., B. PAED.
Head of the Mathematics Department, Danforth Technical School, Toronto	**WALLACE J. MORRISON** B.A., B. PAED.

OF CANADA LIMITED · TORONTO

Copyright, Canada, 1952, 1954

by

THE MACMILLAN COMPANY OF CANADA LIMITED

All rights reserved—no part of this book may be reproduced in any form without permission in writing from the publisher. Mimeographing or reproducing mechanically in any other way passages from this book without the written permission of the publisher is an infringement of the copyright law.

REVISED EDITION, 1960
ALBERTA EDITION, 1964

PRINTED IN CANADA

LIBRARY
UNIVERSITY OF ALBERTA

Preface

First Edition

Good citizenship implies an ability to appreciate and interpret the quantitative aspects of the world about us. *Mathematics in Practice* aims to develop this ability by the use of interesting problem material arising out of the activities of the home, the shop, and the community. While the text will meet the needs of those students whose formal study of mathematics will end with the Intermediate Division, it will also provide a sound foundation for those who plan to continue the study of mathematics in the Senior Division.

The material effectively presents mathematics as the study of quantitative measures and the relationship between these measures. It thus not only provides a thorough review of the work of previous grades but also stimulates in students a greater appreciation of the meaning of mathematical operations in the practical world of science, commerce and industry. At the same time they will be spurred on to greater effort as they become increasingly conscious of the need for a clear understanding of the computational processes and for accuracy in mechanical operations. Finally, by continuous emphasis on the basic elements common to all quantitative thinking, the text provides a general plan by which students may attack many mathematical problems.

Throughout this text the solution of mathematical problems within the experience of the students is paramount. As each new topic is introduced it is illustrated by problems which require the application of the new principles involved. The initial problems in each exercise provide simple settings in which the newly acquired knowledge functions. These are followed by examples designed to develop skill in using the new principles. The exercises are then concluded with more involved problems illustrating relationships between the new topic and other aspects of problem solving.

The descriptive material and many of the problems will suggest the correlation between mathematics and other subjects. For most students the historical references will have a special interest and will lead them to investigate further the part that mathematics has played in man's progress. These and the examples dealing with such topics as conservation, public revenue and various types of insurance provide for the correlation of

279227

mathematics and social studies. Valuable training in clear thinking and accurate expression is provided by requiring students to make concise statements in the solution of problems. The emphasis upon measurement strengthens the bond between mathematics and science. Shop problems and scale drawing stress the importance of mathematics in technical work. This emphasis on the inter-relationship of mathematics with other areas of study will, it is hoped, help students to integrate their newly acquired knowledge into effective educational units.

Second Edition

In response to requests for material to cover the work of Grade X mathematics in technical schools, *Mathematics in Practice* was enlarged to include sufficient material to cover all the topics generally taken in Grades IX and X of a technical school course.

Third Edition

The third edition of *Mathematics in Practice* maintains the same basic principles for the presentation of mathematics in the Intermediate Division as those set forth in the former editions. Time and classroom experience have indicated that some change in order of presentation of topics should be made and that the scope of the work in algebra and geometry should be broadened. The present edition incorporates these changes.

For added convenience, the exercises are graded. Part A of each exercise provides examples which may be solved orally. These review pertinent material and give practice in the use of the new principles of the lesson. Part B consists of problems of average difficulty. These, in general, will require written solutions. Part C contains more difficult examples.

The authors wish to express their appreciation to the many teachers whose valuable suggestions have contributed so notably to this revision.

Alberta Edition

The additions and changes made in the section of the text, chapters 12-16 inclusive, are mainly the work of William Naciuk, Eastglen Composite High School, Edmonton, Alberta. The authors are appreciative not only of his work but of the recommendations made by members of the Subcommittee on Vocational Mathematics of the Department of Education.

A. E. B.
D. E. B.
W. J. M.

Contents

Chapter		*Page*
1	Introduction to Geometry	*1*
	Use of the Instruments	
	Measurement of a Line	
	Measurement of Angles	
2	Geometric Constructions	*19*
	Classification of Triangles	
	Construction of Triangles	
	Right Bisector of a Line	
	Bisection of an Angle	
	Construction of Perpendiculars	
	Parallel Lines	
3	Integral Measures	*38*
	Powers of Numbers	
	Prime Factors and Square Root	
	Multiples and Lowest Common Multiple	
4	Fractional Measures	*57*
	Fundamental Principle of Common Fractions	
	Addition and Subtraction of Common Fractions	
	Multiplication of Fractions	
	Division of Fractions	
	Complex Fractions	
	Decimal Numeration and Notation	
	Exact and Approximate Measures	
	Multiplication of Decimal Fractions	
	Division of Decimal Fractions	

Chapter *Page*

5 PERCENTAGE *99*

Comparisons Expressed as Per Cent
Percentage
Take Home Pay
Cost of Shelter
Per Cent in Retail Business
Commission

6 MEASUREMENT OF DISTANCE *111*

English Linear Units of Measurement
Specialized Linear Units of Measurement
Metric Linear Units of Measurement
Measure Expressed by a General Number
Perimeters of Rectangles and Squares
Circumference of a Circle

7 THE MEASUREMENT OF AREA *127*

Area of a Rectangle and Square
Square Root
English Square Units of Measurement
Metric Square Units of Measurement
Area of a Parallelogram
Area of a Triangle
Area of a Circle
Right-Angled Triangle Relationship

8 MEASUREMENT OF VOLUME *162*

Volume of a Rectangular Solid
English Cubic Units of Volume
Special Units of Volume
English Units of Capacity and Weight
Metric Units of Volume, Weight and Capacity
Cube Root
Volumes of Prisms
Cylinders

9 GENERAL NUMBER *185*

Indicated Operations
Changing the Form of a Measure
Variables and Constants
Substitution
The Language of General Number
Simplification of Expressions

Chapter *Page*

10 SIGNED NUMBERS *197*
Directed Measurements
Signs of Quality and Signs of Operation
The Signed Number System
Line Representation of Signed Numbers

11 OPERATIONS WITH SIGNED NUMBERS *203*
Addition of Signed Numbers
Subtraction of Signed Numbers
Removal of Brackets Preceded by a Plus Sign
Removal of Brackets Preceded by a Minus Sign
Multiplication of Signed Numbers
Division of Signed Numbers
Substitution with Negative Numbers

12 EQUATIONS *230*
Axioms
Solution of Simple Equations
Formulas
Algebraic Method of Solving Problems

13 OPERATIONS WITH ALGEBRAIC EXPRESSIONS *243*
Monomials, Binomials, and Trinomials
Multiplication by a Monomial
Multiplication by Binomials
Square of a Binomial
Product of Sum and Difference
Division by a Monomial

14 FACTORING *257*
Common Factor
Trial Trinomials
Complete Square
Difference of Squares
Applications of Factoring

15 EQUATIONS, FORMULAS AND PROBLEMS *275*
Transposition of Terms
Equations Requiring Simplification
Equations with Fractional Coefficients
Cross Multiplication

Chapter *Page*

16 Graphs and Simultaneous Equations *288*
Relationships expressed by Table of Values, Formula, Graph
System of Cartesian Co-ordinates
Graphs of Algebraic Equations
Simultaneous Equations Solved Graphically
Simultaneous Equations Solved Algebraically

17 Scales and Scale Reading *301*
Commonly Used Scale Division Systems
Approximate Fractional Equivalents
Micrometer and Vernier Scales

18 Shop Calculations *312*
Cutting Speeds—Grinder, Milling Machine, Lathe
Feed-Lathe, Drill Press
Ratio
Direct and Inverse Proportion
Gear and Pulley Speeds
Taper

19 Geometrical Constructions and Relationships *338*
Angles
Properties of Triangles
Inscribed and Circumscribed Circles
Sum of the Angles of Polygons
Construction and Properties of Regular Polygons
Parallel Lines
Division of a Line into Equal Parts
Circle—Segment, Sector, Tangent

20 Mensuration of Triangles *357*
Area of a Right-Angled Triangle
Area of an Isosceles Triangle
Altitude and Area of an Equilateral Triangle
Area of any Triangle by the S-Formula
Capacity of Pipes
Diagonal of a Square
Length of the Arc of a Sector of a Circle
Area of a Sector of a Circle

Chapter *Page*

21 The Cone, Pyramid and Sphere *370*
Area and Volume of a Cone
Area and Volume of a Pyramid
Area and Volume of a Sphere

22 Trigonometry *380*
Definition of the Three Principal Ratios
Sine Ratio Used to Solve Right-Angled Triangles
Area of a Triangle Given Two Sides and a Contained Angle
Cosine Ratio Used to Solve Right-Angled Triangles
Tangent Ratio Used to Solve Right-Angled Triangles
Angles of Elevation, Depression and Slope

Table of Trigonometric Ratios *404*

Miscellaneous Tables *405*

Answers *408*

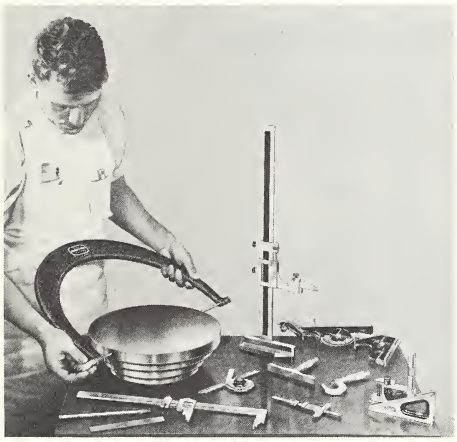
CHAPTER 1

Introduction to Geometry

The Beginning of Geometry

The universe around us has aroused the curiosity of every generation. Since the beginning of time, man has not ceased to wonder about the sun, the moon, the stars and the earth. To satisfy this curiosity and to better provide for his needs, man has learned to observe and measure the things about him. He has compared and related the measures in order to add to his knowledge and to extend control over himself and his surroundings.

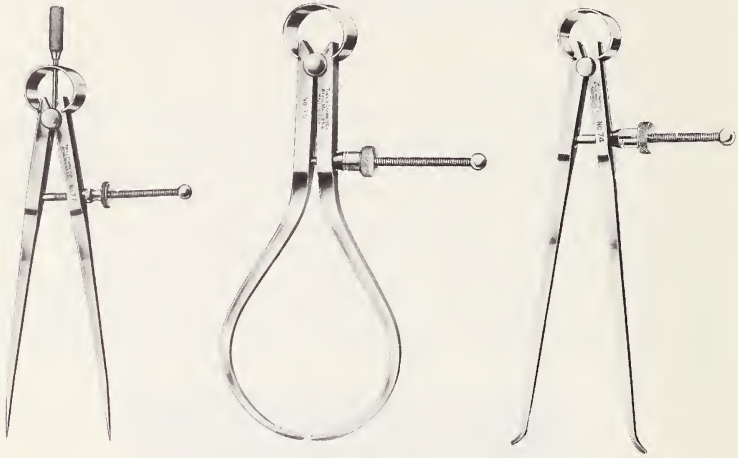
The need for measuring and laying out plots of land gave rise to the beginnings of geometry. For over 2,000 years the science of geometry has provided mankind with the techniques for measuring and calculating distances, angles, areas and volumes.



Courtesy The L. S. Starrett Co.

Precision instruments have been invented to help skilled workers make accurate measurements and draw up detailed plans. Your mathematical set includes a few simple, useful instruments which can be of help to you in the study of geometry.

The compasses are used not only to draw circles but also to locate and mark points at specified distances apart. Dividers are used to transfer straight line distances so that they may be more accurately compared with the scale on a ruler. Calipers are used in the shops for this purpose.



Courtesy The L. S. Starrett Co.

Dividers **Outside Caliper** **Inside Caliper**

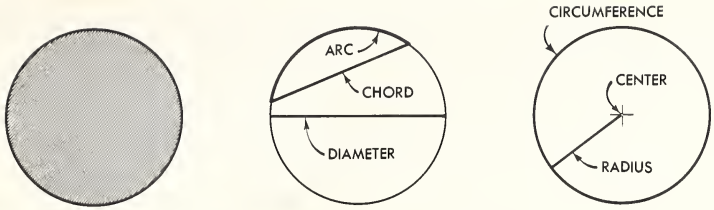
Measuring with Precision

The ruler is used as a straight edge to draw straight lines and as a graduated scale to obtain measures of distance. One scale on your ruler is graduated in inches and fractional parts of an inch. The other scale is divided into centimetres and millimetres. You will notice that one centimetre is a little less than $\frac{1}{2}$ inch.

Straight and Curved Lines

Distances may be measured along straight or curved lines. Usually, unless otherwise specified, lines are considered to be straight lines and may be considered as far as position and direction are concerned to continue indefinitely in both directions. In practice we work with parts of lines or *line segments.*

The simplest curved line is that of the circle. Parts of the circle with which we shall be concerned include examples of straight and curved lines.



The Circle and Its Parts

A **circle** is a plane figure enclosed by a line each point of which is the same distance from a fixed point, the **centre.**

The **circumference** is the curved line forming the boundary of the circle.

A **radius** is the distance from the centre to the circumference.

An **arc** is a portion of the circumference.

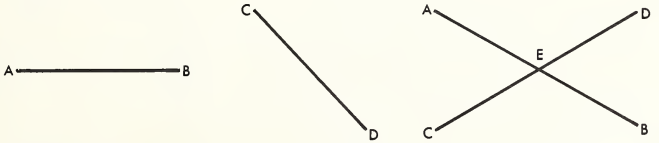
A **chord** is a straight line joining any two points on the circumference.

A **diameter** is a chord which passes through the centre; it is equal in length to twice the radius.

Naming Points and Straight Lines

The position of a point is best marked by drawing two lines to cross each other at the desired position. The point is named by placing a capital letter near it.

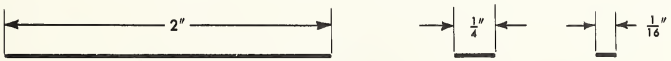
Lines are identified by placing capital letters at two points along the line, usually at the ends.



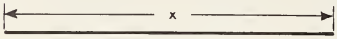
Line AB **Line CD** **Line AB intersects line CD at point E.**

To Indicate the Length of a Given Line

When the lengths of straight lines are to be indicated on a diagram, the measure of the line and the unit of measurement are shown by means of a dimension line. The ends of the arrow-heads indicate the end of the segment.

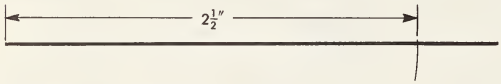


A small letter is sometimes used to indicate the measure of the length of a line.



To Construct a Line of Given Length

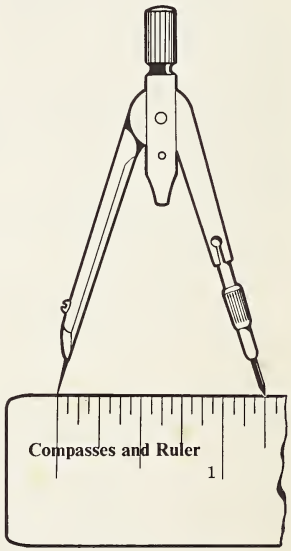
With a straight edge draw a line longer than is required. Adjust the points of the compasses to the required distance. With this radius and centre at one end of the line draw an arc to cut the line.



To Measure a Given Line

To obtain the measurement of the length of the line set the points of the dividers or compasses at a distance apart equal to the length of the line, and transfer this distance to the ruler where the length can be ascertained.

Some rulers are especially designed so that reasonably good results are obtained without the use of the compasses or dividers. With equal care, however, more accurate results can always be obtained by using the compasses, dividers, or calipers to transfer the distance from the line or object to the ruler.



Exercise

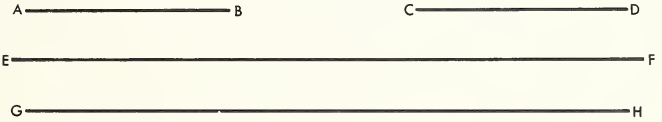
A

1. Explain how to construct accurately a line $3\frac{1}{4}''$ in length.
2. Study the metric scale on your ruler. How many spaces are there between two centimetre graduations? Each of these small divisions is called a millimetre. How many millimetres are there in 1 centimetre? What fraction of a centimetre is each millimetre? The abbreviation for centimetre is cm.; for millimetre, mm.

3. When a ruler was placed on a certain line **AB** it was found that it measured 6 complete centimetre units and 3 small spaces or millimetre units. What is the measure of this line in centimetres? What is the measure of this line in millimetres?
4. What purpose is served by the arrow-heads at the ends of a dimension line?

B

5. Draw a line $3\frac{3}{4}''$ long. Letter and dimension it.
6. Draw a line **ST**, $4\frac{3}{8}''$ long. Find by measurement the middle point **E** and correctly dimension the line and its parts. With your compasses check your accuracy by comparing the lengths **SE** and **ET**.
7. Draw a line **XY** 5″ long. Find points **A** and **B** in the line so that **XA** = $2\frac{3}{8}''$ and **YB** = $1\frac{1}{4}''$. *Calculate* the distance **AB** and record it on your diagram.
8. Draw a line $4\frac{3}{8}''$ long and from one end of the line mark off equal segments, each $\frac{9}{16}''$. How many complete pieces did you get? What length is left? Check the accuracy of your drawing by arithmetical calculation.
9. Measure these four lines to the nearest 16th of an inch and record your measurements. Find the total length of these four lines if they were put end to end.



10. Draw and dimension lines of 3·5 cm., 6·7 cm., and 8 mm.
11. Construct the following lines accurately, **AB** = 2″, **CD** = 3″, **EF** = 4″, **GH** = 5″ and **MN** = 6″. As accurately as possible, measure each line in centimetres. Record your results in a table similar to the following. For each line calculate the number of centimetres in 1 inch and complete the table.

Name of line	*Length in inches*	*Length in centimetres*	*Number of centimetres in 1 inch*

Find your average for the number of centimetres in 1 inch.
The correct number of centimetres in 1 inch is 2·54.

12. Find the number of centimetres in the length of each of the lines with lengths as follows:

(*a*) 8 in., (*b*) 9 in., (*c*) 15 in., (*d*) 2 ft., (*e*) 1 yd.

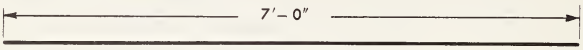
13. Find the number of inches in each of the following lengths:

(*a*) 8 cm., (*b*) 10 cm., (*c*) 100 cm., (*d*) 3 cm.

Express these results correct to 2 places of decimals.

Scale Drawing

If the actual distance to be represented is too long to be drawn on the paper it may be *drawn to scale.* The scale should be indicated and the actual length marked on the diagram.



Scale $\frac{1}{2}'' = 1'\,0''$

The length of this scale drawing, in inches, is $7 \times \frac{1}{2} = 3\frac{1}{2}$.

Here $\frac{1}{2}''$ on the drawing represents 1 foot on the original line. This would be expressed on the drawing in some abbreviated form such as, Scale $\frac{1}{2}'' = 1'\,0''$.

In drafting, small parts are often drawn to full scale or even enlarged. A double scale would be indicated as $\frac{2}{1}$. The scale would be chosen so that the drawing could be put on a page of suitable size and made large enough that details could be shown on the diagram.

Exercise

A

1. What length of line on a scale drawing will represent each of the following distances if the scale $3'' = 1'\,0''$ is used:

(*a*) 2 ft., (*b*) 18″, (*c*) 6″, (*d*) 8″, (*e*) 2″?

2. Using the scale $\frac{2}{1}$ what length on the drawing will represent each of the following lengths:

(*a*) $\frac{3}{8}''$, (*b*) $\frac{1}{4}''$, (*c*) 3″, (*d*) $\frac{3}{16}''$, (*e*) $\frac{5}{8}''$?

3. If the scale is $\frac{4}{1}$, what lengths are represented by lines on the drawing with lengths measuring:

(*a*) $\frac{1}{2}''$, (*b*) $\frac{3}{4}''$, (*c*) $1\frac{1}{2}''$, (*d*) $\frac{1}{8}''$?

4. Using the scale $\frac{3}{8}'' = 1'\,0''$, what lengths on a drawing would indicate the following measurements:

(*a*) 4 ft., (*b*) 12 ft. 6 in., (*c*) 40 ft., (*d*) 6 in.?

5. When the scale $\frac{1}{2}'' = 1'\,0''$ is used in making a drawing, what would be

the actual distances represented by lengths as follows:

(*a*) 3″, (*b*) $2\frac{1}{4}$″, (*c*) $3\frac{1}{8}$″, (*d*) $5\frac{1}{2}$″?

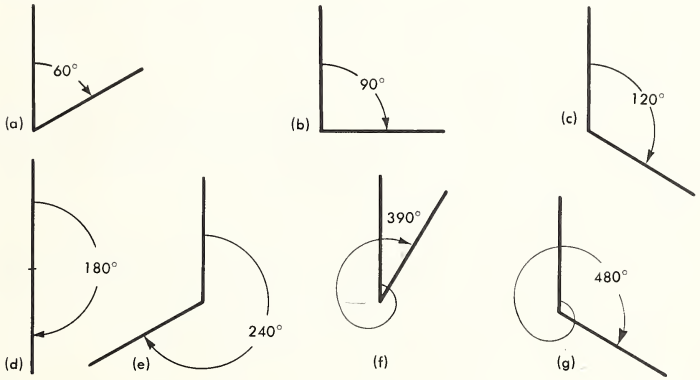
6. When the scale $\frac{3}{16}'' = 1'\,0''$ is used, what length on a scale drawing would represent each of the following:

(*a*) 30 ft., (*b*) 8 ft. 6 in., (*c*) 60 ft. 4 in.?

7. A scale drawing of a rectangle is $3\frac{1}{2}$″ by $2\frac{1}{4}$″. If the scale $2'' = 1'\,0''$ was used, what would be the area of the actual rectangle?

Angles

Each of the diagrams below indicates an amount of turning of a line about a fixed point. The line at the beginning of the dimension arc is considered to be the *initial* position; the line at the end of the arrow-head the *final* position.



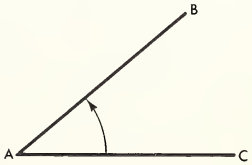
An angle is the amount of turning of a line about a fixed point.

Each of the lines used is called an **arm** of the angle. The point at which the arms meet is the **vertex** of the angle.

In surveying or in navigation an angle may be used to indicate a difference in direction.

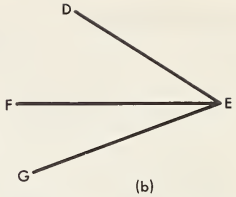
Naming Angles

The letters used in naming the arms serve also to name the angle. The angle in the diagram (*a*) right, may be called angle **A**, abbreviated form $\angle$**A**. If there were more than one angle with vertex **A**, we would name this angle $\angle$**BAC** or $\angle$**CAB**.



(a)

The letter at the vertex is placed between the other two letters. In the diagram (*b*) right, since the point **E** is the vertex for at least two angles, it is necessary to use three letters to name each angle.



(b)

Angular Units of Measurement

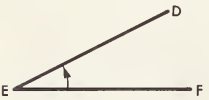
The size of the angle is the amount of rotation of one line from the other about a common point. The basic unit of turning is 1 revolution. For angles less than 1 revolution the derived units of degree, minute and second are used.

1 revolution = 360 degrees
1 degree (1°) = 60 minutes (60′)
1 minute (1′) = 60 seconds (60″).

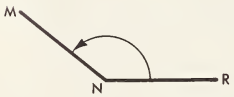
Classification of Angles

A **right angle** is an angle of 90 degrees or $\frac{1}{4}$ of a revolution. ∠**ABC** is a right angle.

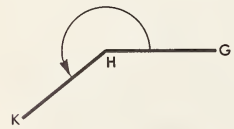
A **straight angle** is an angle of 180 degrees or $\frac{1}{2}$ of a revolution. ∠**PQR** is a straight angle.



An **acute angle** is an angle less than 90 degrees. ∠**DEF** is an acute angle.



An **obtuse angle** is an angle greater than a right angle but less than a straight angle. ∠**MNR** is an obtuse angle.



A **reflex angle** is an angle that is greater than a straight angle. ∠**KHG** as marked is a reflex angle.

Addition and Subtraction of Angles

Since angles are measurable quantities, it is possible and often necessary to find the sum or difference of two angles.

Example 1: If angle **ABC** is 150° and angle **DEF** is 83° find:

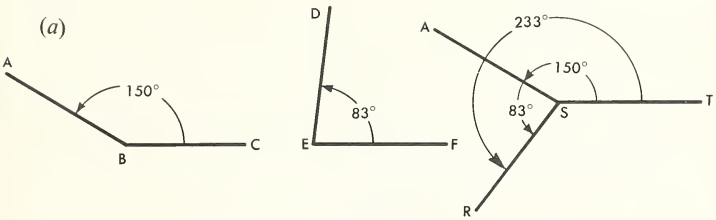
(*a*) the sum of $\angle$**ABC** and $\angle$**DEF**

(*b*) the difference between $\angle$**ABC** and $\angle$**DEF**

(*c*) one half the sum of $\angle$**ABC** and $\angle$**DEF**.

Solution:

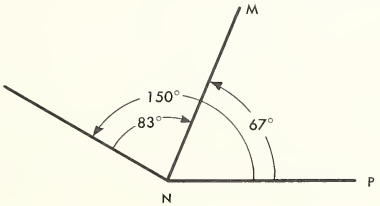
(*a*)



$$\angle\mathbf{ABC} + \angle\mathbf{DEF} = 150° + 83° = 233°$$

In the diagram the reflex angle **RST** = 233°.

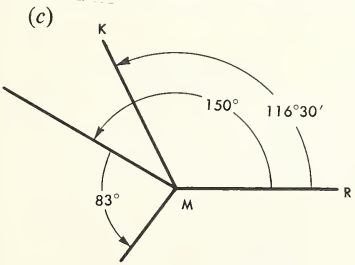
(*b*)



$$\angle\mathbf{ABC} - \angle\mathbf{DEF} = 150° - 83° = 67°$$

As shown in the diagram $\angle$**MNP** = 67°.

(*c*)



$$\angle\mathbf{ABC} + \angle\mathbf{DEF} = 233°$$

$$\tfrac{1}{2}(\angle\mathbf{ABC} + \angle\mathbf{DEF}) = \frac{233°}{2}$$

$$= \frac{232° 60'}{2}$$

$$= 116° 30'$$

As shown in the diagram $\angle$**KMR** = 116° 30′.

Example 2: The sizes of two given angles are 80° 23′ 32″ and 29° 42′ 35″. Find (*a*) their sum, (*b*) their difference, (*c*) $\frac{1}{2}$ their difference.

Solution:

(*a*)

$$\begin{array}{rr} & 80^\circ\ 23'\ 32'' \\ & 29^\circ\ 42'\ 35'' \\ \hline \text{Sum is} & 109^\circ\ 65'\ 67'' \\ \text{Sum is} & 109^\circ\ 66'\ \ 7'' \\ \text{Sum is} & 110^\circ\ \ 6'\ \ 7'' \end{array}$$

(*b*)

$$\begin{array}{r} 80^\circ\ 23'\ 32'' \\ 29^\circ\ 42'\ 35'' \\ \hline \end{array}$$

To accommodate the subtraction of the seconds, one minute is taken from the 23′ and 60 seconds is added to make 92″. Similarly one degree is changed to minutes. Thus the problem becomes

$$\begin{array}{rr} & 79^\circ\ 82'\ 92'' \\ & 29^\circ\ 42'\ 35'' \\ \hline \text{The difference is} & 50^\circ\ 40'\ 57'' \end{array}$$

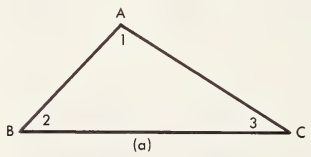
(*c*) One half the difference is $\dfrac{50^\circ\ 40'\ 57''}{2}$

$$= 25^\circ\ 20'\ 27\tfrac{1}{2}''$$

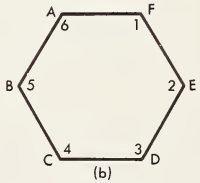
Exercise

A

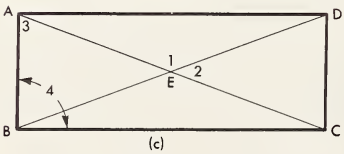
1. Name the angles in the following diagrams in the order in which they are numbered in each diagram:



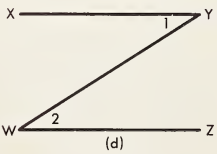
(a)



(b)



(c)



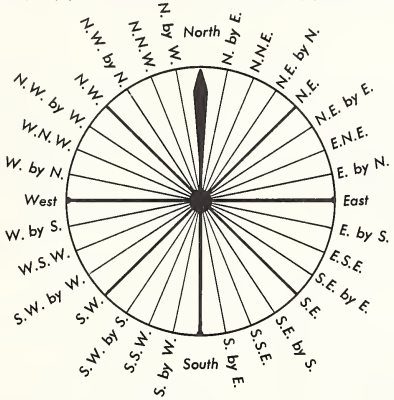
(d)

2. State the size and the kind of angle which is formed when:

(*a*) a horizontal and a vertical line meet,
(*b*) the hands of a clock indicate 2 o'clock, 4 o'clock, 7 o'clock,
(*c*) the front and back covers of an open book are placed flat on a desk,
(*d*) the hour hand of a clock turns through 10 hours,
(*e*) the hands of a clock indicate 3:30.

3. State, in degrees, the difference in direction between

(*a*) east and west, (*b*) north and northeast, (*c*) north and southwest.



4. Starting with the north direction, name the eight main points of the Mariner's compass.

5. What direction is midway between east and southeast?

B

6. Construct a neat sketch of a card for a Mariner's 16-point compass.

7. What is the size of angle through which a ship turns when it changes direction from

(i) north to northeast, (ii) northeast to southeast,
(iii) southwest to west, (iv) west to north-northeast?

8. In aerial navigation the Mariner's compass is being replaced by one in which the directions are expressed in terms of clockwise rotation from the north. Such readings are called bearings. A landmark east of the observer would have a bearing of 90°.

Express each of the following directions as bearings:

(*a*) southeast, (*b*) south,
(*c*) west, (*d*) northwest,
(*e*) north-northwest, (*f*) south-southeast.

9. Express as points on the Mariner's compass each of the following bearings:
(*a*) 45°, (*b*) 90°, (*c*) 135°,
(*d*) 157·5°, (*e*) 202·5°, (*f*) 292·5°.

10. What is the angle between the hands of a clock when the time indicated is:
(*a*) 5:20, (*b*) 7:10, (*c*) 8:30?

11. Find the sum of the following angles:
(*a*) 55° 20′ 10″, 75° 8′ 15″, 83° 7′ 5″, 27° 10′ 17″;
(*b*) 150° 25′ 35″, 75° 15′ 40″, 220° 10′ 35″, 120° 5′ 15″.

12. Find the difference between:
(*a*) 75° 25′ 15″ and 32° 12′ 10″, (*b*) 256° 23′ 5″ and 127° 17′ 40″,
(*c*) 82° 10′ 7″ and 59° 15′ 25″.

13. A pulley rotates with a speed of 200 r.p.m. Through how many degrees per second does it turn?

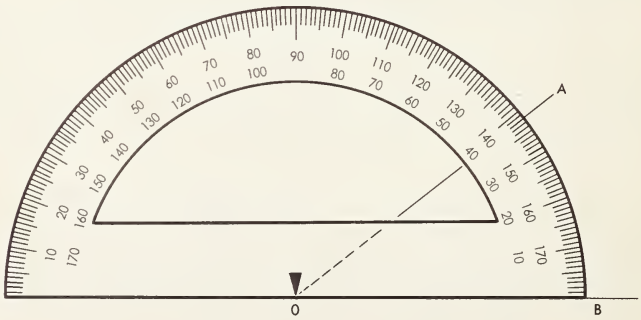
14. Calculate one half of each of the following angles:
(*a*) 26° 40′ 30″, (*b*) 52° 49′ 18″, (*c*) 185° 37′ 36″.

15. Find one half the sum of 28° 32′ 51″ and 95° 40′ 53″.

16. Find one half the difference of (*a*) 45° and 26°, (*b*) 175° 15′ and 83° 40′.

Measurement of Rotation

The instrument for measuring angles is called a protractor. One of the simpler forms of protractor is included in your set of mathematical instruments.



$\angle AOB = 38°$

At the middle point of the base a mark indicates the point on the protractor which corresponds to the vertex of the angle. The large divisions on the

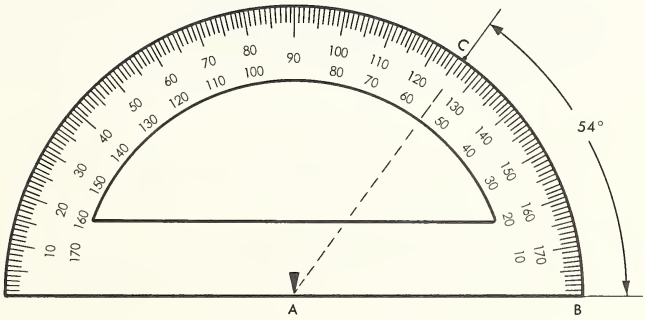
scale indicate intervals of 10 degrees. The outer scale begins at the left and is used for measuring rotation in a clockwise direction. The inner scale begins at the right and is used to measure rotation in a counter-clockwise direction.

When measuring an angle place the base line of the protractor along one arm of the angle with the centre of the protractor at the vertex. If you have done this correctly one arm of the angle will cut the scale at 0°. The measure indicated by the position of the other arm gives the size of the angle in degrees.

Note: The length of the arm of an angle does not affect the size of the angle. If the arm is not long enough for accurate measurement, extend it to the desired size.

Constructing a Given Angle

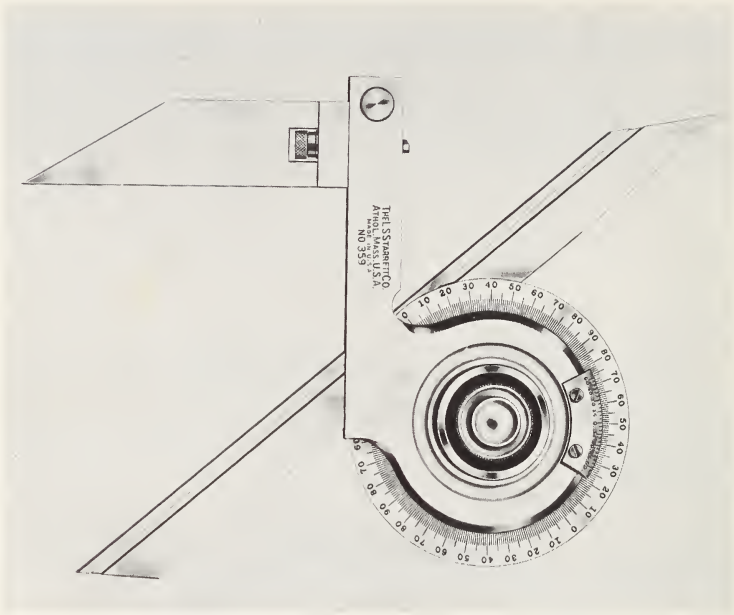
To make an angle of given size, start with a line **AB** where **A** is to be the vertex of the angle. Place the protractor with the centre point at **A** and the base line of the protractor along **AB**. Mark **C** the point on the paper opposite the required measure. From **A** draw a line through **C**. Mark the size of the angle.



Exercise

A

1. State the proper way to name an angle, using three letters.
2. Define (*a*) acute angle, (*b*) obtuse angle.
3. How many right angles are there in (*a*) a straight angle, (*b*) one rotation?
4. What direction on the Mariner's compass is a bearing of 45°, 90°, 180°, 135°?
5. What is the size of the angle between (*a*) NE and NW, (*b*) NNW and N, (*c*) NE and E?
6. What size of angle is usually used in mitred joints?

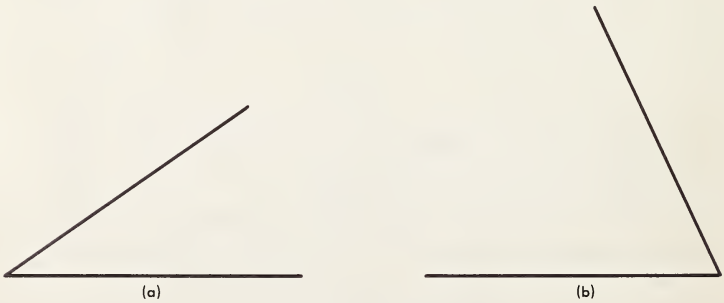


Courtesy The L. S. Starrett Co.

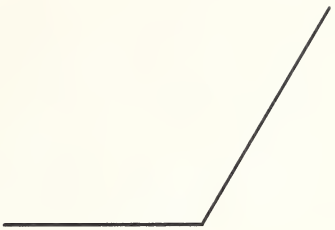
Bevel Protractor

This instrument is used by machinists when an accurate measurement of the angle between two surfaces is required.

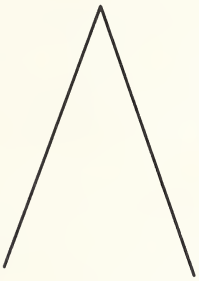
7. Measure these angles with your protractor.



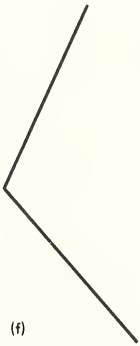
(c)



(d)



(e)



(f)

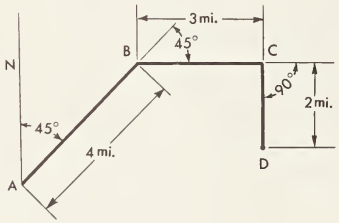
B

8. Construct with your protractor, name, and record the size of each of the following angles:
   (*a*) 28°, (*b*) 45°, (*c*) 60°, (*d*) 150°, (*e*) 110°.
9. Use your protractor and ruler to draw an accurate rectangle **ABCD** three inches long and two inches wide. Measure the angles at which the diagonals intersect at the point **E**. Name and record the size of each angle.
10. Draw two straight lines **AB** and **CD** crossing each other at **E**. Measure and record the size of each of the four angles.
11. Draw other pairs of straight lines crossing each other. Measure and record the size of each of the four angles formed.
12. In a circle draw any chord **AB**. Take any point **C** on the circumference. Join **CA** and **CB** to form the angle **ACB**. Choose other points **D**, **E**, **F** on the same arc of the circumference. Measure and compare angles **BCA**, **BDA**, **BEA** and **BFA**.

13. Draw a circle and its diameter **AB**. Mark other points **D**, **E**, **F** on the circumference. Compare the size of the angles **ADB**, **AEB** and **AFB**.

14. Draw any circle and name its centre **C**. Draw any chord **MN** of this circle. Take any points **A**, **B**, **E** on the larger arc of the circumference. Compare the size of angle **MCN** with the size of the angles **MAN**, **MBN** and **MEN**.

15. The following is a sketch of a route followed by a scouting party who started from a point indicated by **A** and travelled northeast for 4 miles to **B**, turned east and travelled 3 miles to **C**, then turned south 2 miles to **D**.



Make an accurate scale drawing of this route and find the shortest distance from **A** to **D**.

16. Construct a sketch of the route taken by a scouting party who starting at a point **A**, went 4 miles east, then 2 miles northwest, then 3 miles northeast and finally 1 mile southeast to point **E**. Draw the route to scale. Measure the distance **AE** and calculate the distance they are from **A**.

C

17. From an observation point a scout finds that the spire of a church has a bearing of 56° 35′ from the north. A water tower near the horizon has a bearing of 73° 10′. Calculate the difference in these bearings.

18. From an observation point, landmarks designated as **A**, **B**, **C** and **D** are located as follows: Bearing of **A** is 25° 55′; **B** can be sighted by turning farther to the right through 30° 23′; **C** by turning still farther through 75° 25′ and **D** by turning again to the right through 95° 45′.
    (*a*) State the bearing of each of the points **B**, **C** and **D** from the observation point.
    (*b*) If an observer is facing **D** through what angle will he have to turn in order to face north?

Review Exercise

A

1. State three ways in which your compasses may be used in geometry.
2. Distinguish between a straight line and a line segment.
3. Define the terms: circle, circumference, radius, arc, chord and diameter.
4. Express the following lengths in centimetres:
   (*a*) 1″, (*b*) 10″, (*c*) 5″, (*d*) 20″, (*e*) 5 ft.
5. What length would be used in a drawing with scale: $2'' = 1'\ 0''$ to represent distances as follows:
   (*a*) 5′, (*b*) 3′ 6″, (*c*) 0′ 3″, (*d*) 2′ 9″?
6. A drawing is made to scale: $\frac{1}{2}'' = 1'\ 0''$: What is the actual length of line represented on the drawing by
   (*a*) 2″, (*b*) 1″, (*c*) $\frac{1}{2}''$, (*d*) $4\frac{1}{2}''$, (*e*) $3\frac{1}{2}''$?
7. A drawing is made to scale: $\frac{2}{1}$. What is the actual length of a line which is represented on a drawing by
   (*a*) $\frac{1}{4}''$, (*b*) 1″, (*c*) $\frac{1}{8}''$, (*d*) $\frac{3}{4}''$?
8. Define: angle, vertex, arm, straight angle, right angle, acute angle, obtuse angle, reflex angle.
9. How many degrees are there between the hands of a clock at
   (*a*) 10 o'clock, (*b*) 5 o'clock?
10. Through how many degrees does the minute hand of a clock turn in
    (*a*) 5 minutes, (*b*) 15 minutes, (*c*) $\frac{1}{2}$ hour,
    (*d*) 1 hour 10 minutes, (*e*) 1 hour 30 minutes?
11. What is the size of angle between the directions
    (*a*) N and S, (*b*) NE and SE, (*c*) NE and SW, (*d*) NE and SE?
12. Express the following in minutes:
    (*a*) 5°, (*b*) 10° 15′, (*c*) 2° 10′ 30″.
13. How many degrees in
    (*a*) 240 minutes, (*b*) 330 minutes, (*c*) 7,200 seconds?

B

14. Express each of the following measurements in centimetres:
    (*a*) 8 inches, (*b*) 1 foot, (*c*) 2 feet 6 inches, (*d*) 1 yard,
    (*e*) 39·37 inches.
15. Express each of the following measurements in inches:
    (*a*) 10 cm., (*b*) 100 cm., (*c*) 1 cm.,
    (*d*) 5 cm., (*e*) 1 mm., (*f*) 1·5 cm.

16. A floor is represented on a drawing by a rectangle $7\frac{1}{2}''$ by $9\frac{3}{4}''$. If the drawing was made to scale: $\frac{3}{16}'' = 1'\ 0''$, find the perimeter of the floor.

17. A pulley rotates with a speed of 40 r.p.m. Through what angle does it turn in

(*a*) 1 minute, (*b*) 5 minutes, (*c*) $\frac{1}{4}$ second?

18. A survey party starting from headquarters at **A** travelled 3 miles south-east to **B**, then west 8 miles to **C**, then 5 miles north to **D** and then back home. Make an accurate scale drawing to represent this trip and find

(i) the total length of the journey,

(ii) measure the angle **ADC** and hence determine the direction travelled from **D** to **A**.

19. The angle **A** is $70°\ 35'\ 56''$.

(*a*) What size is an angle which is twice the size of angle **A**?

(*b*) What is the size of an angle which is one half the size of angle **A**?

20. The angle **A** is $73°\ 22'\ 15''$ and the angle **B** is $25°\ 32'\ 45''$.

(*a*) Find an angle which is one half the sum of angle **A** and angle **B**.

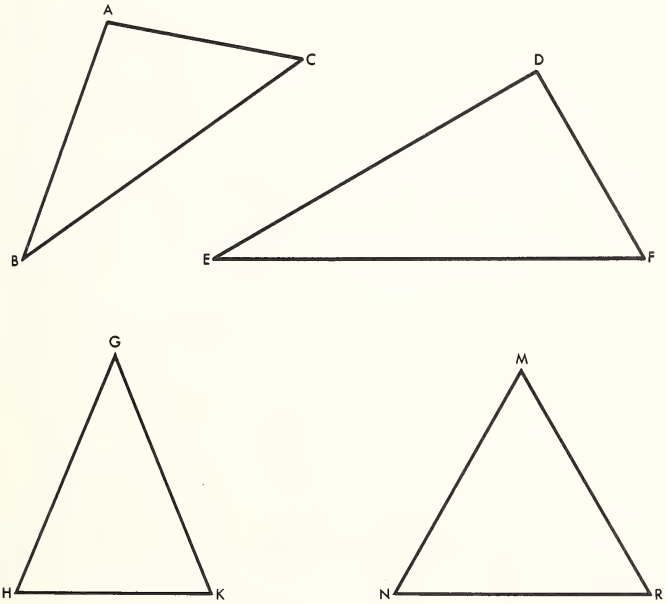
(*b*) Find an angle which is one half the difference of angle **A** and angle **B**.

CHAPTER 2

Geometric Constructions

Triangles

A **triangle** is a plane figure enclosed by three straight lines. The lines form the arms of three angles and enclose a definite area.



Each triangle is named by using *capital* letters placed at the vertices.

Exercise

A

1. Name each of the triangles above in at least three different ways.
2. Name each angle of each triangle in at least two different ways.

B

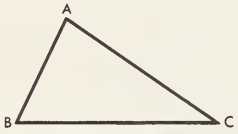
3. In your work book, draw four triangles and measure each angle. Record your results and find the sum of the angles of each triangle. What conclusion is suggested by your results?

Observation

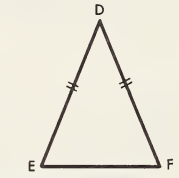
The sum of the three angles of a triangle is 180°.

Classification of Triangles

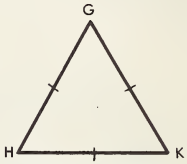
(*a*) **According to the lengths of the sides**



Scalene Triangle



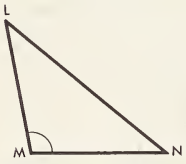
Isosceles Triangle



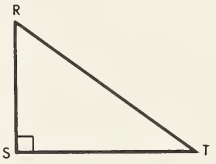
Equilateral Triangle

A **scalene triangle** is one in which all three sides are of different lengths.
An **isosceles triangle** is one in which two sides are equal.
An **equilateral triangle** is one in which all sides are equal.

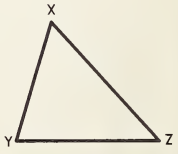
(*b*) **According to the sizes of the angles**



Obtuse-angled Triangle



Right-angled Triangle



Acute-angled Triangle

An **obtuse-angled triangle** is one in which one of the three angles is an obtuse angle.

A **right-angled triangle** is one in which one of the three angles is a right angle.

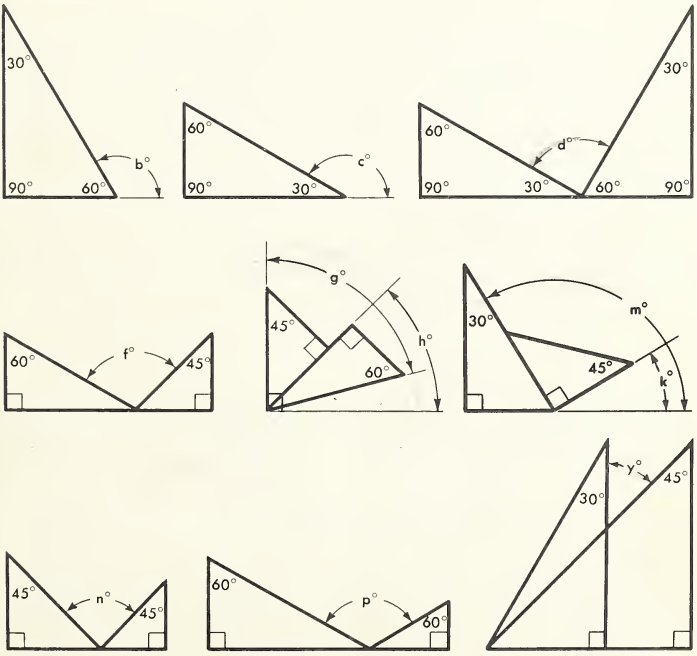
An **acute-angled triangle** is one in which all three angles are acute.

Exercise

A

1. Distinguish between an obtuse angle and an obtuse-angled triangle.
2. Define a scalene triangle; an isosceles triangle; an equilateral triangle.

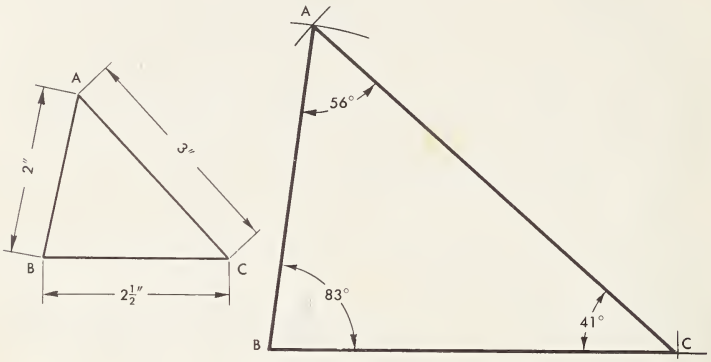
3. In a right-angled triangle one acute angle is 50°. What is the size of the third angle?
4. Explain why it is necessary to define an acute-angled triangle as having three acute angles.
5. If a triangle has two angles as follows, what is the size of the third angle:
   (*a*) 60°, 100°; (*b*) 45°, 45°;
   (*c*) 90°, 30°; (*d*) 110°, 25°;
   (*e*) 30° 20′, 75° 15′; (*f*) 53° 47′ 10″, 80° 25′ 53″?
6. If triangle **ABC** has $\angle \mathbf{A} = 50°$ and $\angle \mathbf{B} = 75°$, find the size of $\angle \mathbf{C}$.
7. In triangle **DEF** $\angle \mathbf{D}$ is 65° and $\angle \mathbf{E} = \angle \mathbf{F}$. Find the size of $\angle \mathbf{E}$.
8. If the three angles of a triangle are equal in size, what is the size of each?
9. If the sum of two angles of a triangle is 90°, what is the size of the third angle?
10. If one of the acute angles of a right-angled triangle is $35\frac{1}{2}°$, what is the size of the other acute angle?
11. Find the number of degrees represented by the letters in the following diagrams:



Construction of Triangles

To construct a triangle when the lengths of the three sides are known.

Example: Construct a triangle **ABC** for which the following measurements are given: **AB** is 2″, **BC** is $2\frac{1}{2}$″ and **AC** is 3″. By measuring determine the sizes of the other parts of the triangle.



Method: Draw a neat sketch and mark on it the given dimensions.
Draw a line **BC** $2\frac{1}{2}$″ long.
With centre **B** and radius 2″ draw an arc above the line.
With centre **C** and radius 3″ draw an arc cutting the former arc at **A**.
Join **AB** and **AC**.
ABC is a triangle in which **AB** = 2″, **BC** = $2\frac{1}{2}$″ and **AC** = 3″.
By measuring it is found that ∠**A** is 56°, ∠**B** is 83° and ∠**C** is 41°.

The lengths of the sides of a triangle determine the sizes of all the other parts of the triangle.

Exercise

B

In all cases when accurate drawings are required make a neat sketch to illustrate the required figure. Letter it and mark the given dimensions in their proper places. Use this sketch for reference when making the accurate drawing.

1. Construct triangle **ABC** having **AB** = 3″, **BC** = $3\frac{1}{4}$″ and **CA** = $3\frac{1}{2}$″. Describe your method. Measure and record the size of each angle.
2. Construct triangle **GHK** with **GH** = 3″, **HK** = 3″ and **GK** = 5″. Describe your method. Measure and record the size of each angle.
3. Draw a triangle with sides of 3″, 4″ and 5″. Describe your method. Measure and record the size of each angle.

4. Construct a triangle each side of which measures 4″. Measure and record the size of each angle.

5. Classify the triangles you have drawn and record your classification in a table as follows:

Name of triangle	*Classification according to*	
	Angles	*Sides*

6. Draw three different isosceles triangles. Measure and record on your diagram the size of each of the angles. State a conclusion suggested by your results.

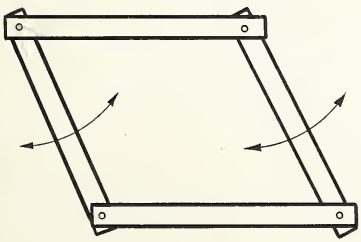
7. From your conclusion in question 6, what size of angle will you expect to find in an equilateral triangle? Draw three equilateral triangles and check your conclusion by measuring each angle.

8. The dimensions for a triangle **DEF** were given as follows: **DF** = 5″, **EF** = $1\frac{1}{2}$″ and **DE** = 3″. Try to construct a triangle using these measurements. Suggest a relationship that exists with respect to the lengths of the sides of a triangle.

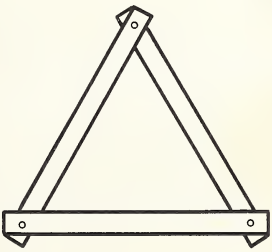
9. Construct two different four-sided figures—quadrilaterals. Find the sum of the angles of each. What conclusion is suggested?

C

10. Construct a figure **ABCD** having the lengths of its four sides as follows: **AB** = 2″, **BC** = 3″, **CD** = $2\frac{1}{2}$″ and **DA** = $2\frac{1}{4}$″. Is it possible to construct different four-sided figures using these specifications? Do the lengths of the sides of a four-sided figure determine the sizes of all the other parts of the figure?

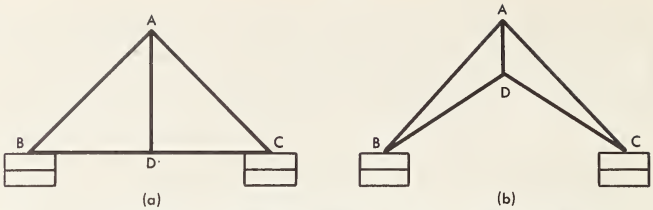


The individual parts are movable. This frame is not rigid.



The individual parts are not movable. This frame is rigid.

11.

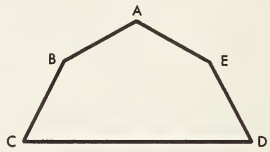


The above diagrams represent a section of two types of framework used for supporting roofs.

(*a*) Explain why these are *rigid* frameworks.

(*b*) Would either or both of these structures be rigid if the bar **AD** were removed? Explain.

12. The diagram at the right shows a section of a type of framework for a roof. Could this be considered a rigid framework? Why?

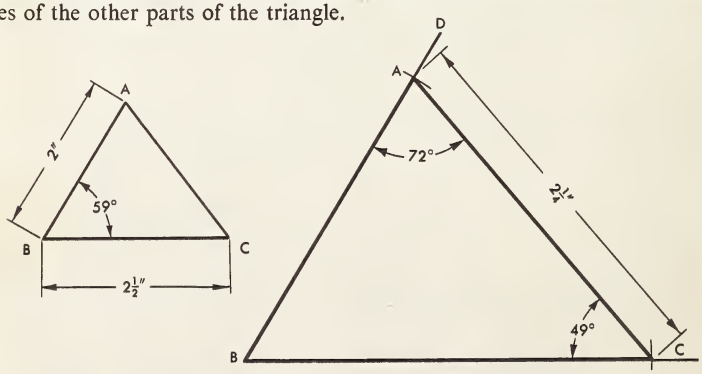


Summary

1. **The sum of the angles of a triangle is 180°.**
2. **The sum of any two sides of a triangle is greater than the length of the third side.**
3. **In an isosceles triangle, the angles opposite the equal sides are equal.**
4. **In an equilateral triangle, each angle is 60°.**

To construct a triangle when the measures of two sides and the angle contained by these sides are known.

Example: Construct a triangle **ABC** for which the following measurements are given: **AB** is 2″, ∠**B** is 59° and **BC** is $2\frac{1}{2}$″. By measuring determine the sizes of the other parts of the triangle.

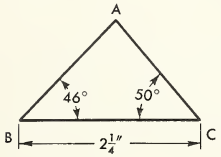


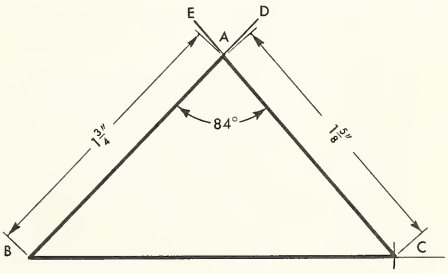
Method: Draw a neat sketch and mark on it the given dimensions.
Draw a line **BC** $= 2\frac{1}{2}''$.
At **B** make $\angle$**CBD** $= 59°$.
From **BD** cut off **BA** $= 2''$.
Join **AC**.
ABC is a triangle in which **AB** $= 2''$, $\angle$**B** $= 59°$ and **BC** $= 2\frac{1}{2}''$.
By measuring, it is found that **AC** $= 2\frac{1}{4}''$, $\angle$**A** $= 72°$ and $\angle$**C** $= 49°$.

The lengths of two sides of a triangle together with the size of the angle contained by these sides determine the size of all the other parts of the triangle.

To construct a triangle when the measures of one side and two angles are known:

Example: Construct triangle **ABC** for which the following measurements are given: **BC** is $2\frac{1}{4}''$, $\angle$**B** is 46° and $\angle$**C** is 50°. By measuring determine the sizes of the other parts of the triangle.





Method: Draw a neat sketch and mark on it the given dimensions.
Draw the side **BC** $= 2\frac{1}{4}''$.
At **B**, make $\angle$**CBD** $= 46°$.
At **C**, make $\angle$**BCE** $= 50°$.
Produce **BD** and **CE** to intersect at **A**.
ABC is a triangle in which **BC** $= 2\frac{1}{4}''$, $\angle$**B** $= 46°$ and $\angle$**C** $= 50°$.
By calculation it is found that $\angle$**A** $= 84°$.
By measuring it is found that $\angle$**A** $= 84°$, **AB** $= 1\frac{3}{4}''$ and **AC** $= 1\frac{5}{8}''$.

If the size of $\angle$**A** and of $\angle$**B** of this triangle had been given, the size of angle **C** could have been calculated and the triangle could have been constructed as described above.

The length of one side of a triangle together with the sizes of two angles determine the sizes of all the other parts of the triangle.

Exercise

A

1. If two sides of a triangle are 14″ and 6″, what is the greatest length of the third side?
2. What is the sum of the angles of a rectangle; a square?
3. Describe the method of constructing triangle **ABC** with sides of 3″, 4″ and 5″.
4. Describe the method of constructing an isosceles triangle with equal sides of 4″ and a contained angle of 30°.
5. Describe the method of constructing triangle **DEF** with $\angle \mathbf{D} = 60°$, $\angle \mathbf{E} = 50°$ and $\mathbf{DF} = 4''$.

B

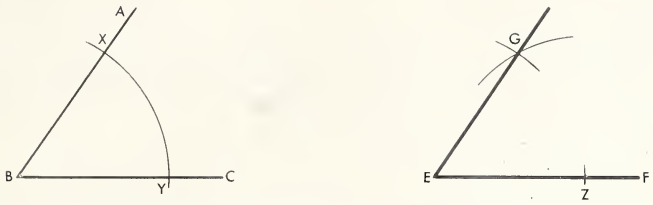
In each of the following problems construct the triangle according to the specifications given; measure and mark on the accurate diagram the dimensions of the other three parts.

6. Construct a triangle **ABC** in which $\mathbf{AB} = 3''$, $\angle \mathbf{B} = 57°$ and $\mathbf{BC} = 3\frac{1}{2}''$. Describe the method.
7. Construct a triangle **ABC** in which $\angle \mathbf{A} = 110°$, $\mathbf{AB} = 2\frac{7}{8}''$ and $\mathbf{CA} = 3\frac{1}{4}''$.
8. Construct a triangle **DEF** in which $\angle \mathbf{E} = 50°$, $\mathbf{EF} = 3\frac{1}{2}''$ and $\angle \mathbf{F} = 70°$. Describe the method.
9. Construct a triangle **GHK** in which $\angle \mathbf{G} = 40°$, $\angle \mathbf{H} = 70°$ and $\mathbf{HK} = 8{\cdot}5$ cm.
10. Construct a triangle **RST** in which $\angle \mathbf{R} = 80°$, $\angle \mathbf{S} = 45°$ and $\mathbf{ST} = 3''$.
11. On the opposite sides of a line **AB** draw two different isosceles triangles **CAB** and **DAB** respectively. Join **CD** to cross **AB** at **E**. Measure and compare:
    (i) **AE** and **EB**;
    (ii) $\angle$**ACD** and $\angle$**BCD**;
    (iii) each of the angles at **E**.

C

12. Make a suitable scale drawing and find the sizes of the remaining parts of each of the following triangles:
    (*a*) Triangle **ABC** in which $\mathbf{AB} = 75'$, $\mathbf{BC} = 60'$ and $\mathbf{AC} = 90'$;
    (*b*) Triangle **DEF** in which $\mathbf{DE} = 8$ miles, $\angle \mathbf{D} = 110°$ and $\mathbf{DF} = 10$ miles;
    (*c*) Triangle **RST** in which $\angle \mathbf{R} = 50°$, $\angle \mathbf{S} = 75°$ and $\mathbf{ST} = 40$ miles.

To Construct an Angle Equal to a Given Angle



Let ∠**ABC** be a given angle and **EF** a given straight line.

It is required to construct at **E** an angle equal to ∠**ABC**.

Method:

With centre **B** and suitable radius draw arcs to cut **AB** at **X** and **BC** at **Y**.

With centre **E** and the same radius draw an arc to cut **EF** at **Z** and extend the arc.

With centre **Z** and radius **XY** draw an arc to cut the former arc at **G**. Draw a line from **E** to pass through **G**.

∠**GEF** is equal to ∠**ABC**.

Exercise

A

1. State a set of three measurements of a triangle which will determine the triangle.
2. How many such sets of measurements are there?
3. State a set of three measurements of a triangle which will not determine a triangle.
4. Explain why the ∠**GEF** in the construction above will have to be equal to ∠**ABC**.

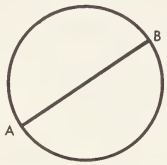
B

5. Draw any obtuse angle **ABC**. Make an angle equal to angle **ABC**. Describe your method.
6. Draw any two acute angles **ABC** and **DEF** respectively. Make an angle which is equal to the sum of angle **ABC** and angle **DEF**.
7. Draw any scalene triangle **ABC**. Make an angle equal to the sum of the three angles of the triangle. What relationship does your construction illustrate?
8. Draw any quadrilateral **ABCD**. Make an angle equal to the sum of the four angles of a quadrilateral. What relationship does your construction illustrate?

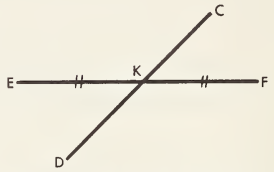
9. Draw any acute angle **ABC**. Mark a point **E** on **AB**. At **E** make an angle **AED** equal to the angle **ABC**. Compare the direction of **ED** and **BC**.

Meaning of Bisect

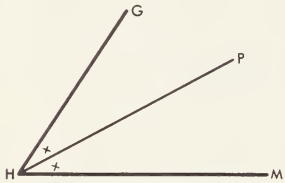
In mathematics, the term *bisect* is used to indicate that a quantity is divided into two equal parts. The line drawn to mark the division is called **the bisector.**



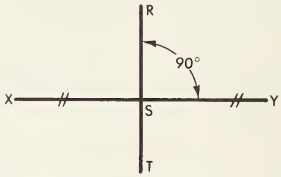
AB *bisects* the area of the circle.



CD *bisects* the line **EF** at **K**.



HP *bisects* the ∠**GHM**.



RT is the *right bisector* of **XY**.

When the bisector of a straight line intersects the original line at right angles it is called the **right bisector**.

To Construct the Right Bisector of a Line

Let **AB** be the given line.

It is required to draw the right bisector of **AB**.

Method: With centre **A** and any radius greater than $\frac{1}{2}$**AB** draw an arc on each side of **AB**.

With centre **B** and the same radius, draw arcs to cut the first pair of arcs at **C** and **D** respectively.

Draw a line through **C** and **D** to cut **AB** at **E**.

CD is the right bisector of **AB**.

Exercise

A

1. State the definition of a right bisector.
2. If **AB** is the right bisector of **CD** does it follow that **CD** is the right bisector of **AB**? Explain.
3. If, in making a path in the snow, a boy always kept the same distance from two fence posts, what path would he follow?

B

4. Draw a line $3\frac{5}{8}''$ long. Find its middle point, using only the compasses and the straight edge. Describe your method.
5. Draw a vertical line **AB**. Construct the right bisector of **AB**. Describe your method.
6. Draw a straight line **AB** at least 4 inches long. Construct its right bisector **XY** to intersect **AB** at **C**. Along **XY** choose 5 points **D**, **E**, **F**, **G** and **H** taken at random. Compare the distances: (i) **DA** and **DB**, (ii) **EA** and **EB**, (iii) **FA** and **FB** etc. How do the distances of each of these points from the ends of the line **AB** compare? Draw the right bisectors of at least three other lines. Check the following statement:

Any point on the right bisector of a straight line is equidistant from the ends of the line.

7. Draw a triangle **ABC** with lengths as follows: **AB** is 6 cm., **BC** is 7 cm. and **AC** is 8 cm. Draw the right bisector of **AB** and the right bisector of **BC**. Produce these bisectors to meet each other at **E**. Compare the distances: (1) **EA** and **EB**, (2) **EB** and **EC**. How should **EA** and **EC** compare? With centre **E** and radius **EA** draw a circle. If your work has been done carefully this circle will pass through **A**, **B** and **C**. Such a circle is said to be **circumscribed** about the triangle. **E** is the **circum-centre** of the triangle.
8. Construct a triangle **MNR** having **MN** $= 1\frac{1}{2}''$, **NR** $= 2''$ and **MR** $= 2\frac{1}{2}''$. By drawing the right bisectors of **MN** and **NR** respectively, locate the circum-centre. Draw the circum-circle. Measure and record its radius.
9. Construct triangle **PQR** having **PQ** $= 2\frac{1}{2}''$, **QR** $= 3\frac{1}{2}''$ and **PR** $= 4\frac{1}{2}''$. Draw and record the measurement of the radius of the circum-circle.

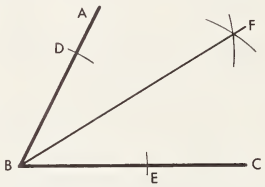
C

10. Construct a circle with centre **O**. Draw a diameter **AB**. Construct the right bisector of **AB** to cut the circle at **C** and **D**. Join **AC**, **CB**, **BD** and **DA**. Measure $\angle$**CAD**. **ABCD** is a square inscribed in a circle.
11. Mark any three points **X**, **Y** and **Z**, not all on the same straight line. Show how we may locate the centre of the circle which passes through **X**, **Y** and **Z**. Draw the circle.
12. Using a coin or some other circular object trace a circle in your work book. Show how to locate the centre of this circle.

To Bisect an Angle

Given $\angle$**ABC**.

It is required to bisect $\angle$**ABC**.



Method: With centre **B** and any radius draw arcs cutting **BA** and **BC** at **D** and **E** respectively.

With centre **E** and radius more than $\frac{1}{2}$**DE**, make an arc between the arms of the angle.

With centre **D** and the same radius, make an arc to cut the former one at **F**.

From **B** draw a line to pass through **F**.

BF is the bisector of the angle **ABC**.

Exercise

A

1. What is the size of one of the angles obtained by bisecting
(*a*) 45°, (*b*) 180°, (*c*) 75°, (*d*) 84° 30′,
(*e*) 27° 30′, (*f*) 13° 40′, (*g*) 67° 30′?

B

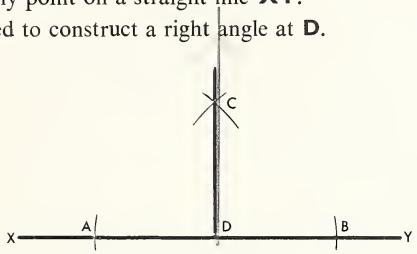
2. Make an angle of 70°. Draw its bisector. Describe your method. Using a protractor, check the size of the two angles.
3. Construct an isosceles triangle. Bisect the vertical angle. At what point does the bisector meet the base? At what angle does it intersect the base?
4. Using the ruler and compasses, divide an angle of 110° into two equal parts. Describe your construction.

5. Draw any triangle **DEF**. With the compasses bisect each angle of the triangle and extend the bisectors until they cross each other.
6. Construct a 16-point card for a Mariner's compass. Use the ruler and compasses to obtain the directions accurately.
7. Draw a straight angle and bisect it.

To Construct a Right Angle at a Point on a Line

Let **D** be any point on a straight line **XY**.

It is required to construct a right angle at **D**.



Method: With centre **D** and any radius, make arcs to cut **XY** at **A** and **B** respectively.

With centres **A** and **B** and a radius longer than **DB**, make two arcs to intersect each other at **C**.

Draw a line from **D** to pass through **C**.

$\angle$**CDY** is a right angle.

Note: The line **CD** is **perpendicular** to the line **XY**.

Exercise

A

1. Explain the difference between bisecting a straight line and bisecting a straight angle.
2. Explain how you would construct an angle of 45° without using a protractor.
3. Explain how to construct an angle of 135° without a protractor.

B

4. A straight line **XY** is drawn and a point **P** is marked on it. With ruler and compasses construct through **P** a line perpendicular to **XY**. Describe your method.
5. Draw a line **AB**. At **B** construct a line perpendicular to **AB**.
6. Draw a line **AB** 4·5 cm. long. Construct a square on the line **AB** as a side. Name the square **ABCD**. Draw the diagonals **AC** and **BD**. Using the protractor, test the lines **AC** and **BD** for perpendicularity. State two relationships that are true about the diagonals of a square.

7. Construct a rectangle **XYZW** with **XY** = 6 cm. and **YZ** = 4·8 cm. Test the diagonals for perpendicularity. State your conclusion.
8. What relationship is true about the diagonals of a square which is not true about the diagonals of a rectangle? What relationship is true about the diagonals of both?
9. Using ruler and compasses construct angles of the following sizes:
   (*a*) 45°, (*b*) 135°, (*c*) 270°,
   (*d*) 22° 30′, (*e*) 67° 30′, (*f*) 225°.
10. Using an equilateral triangle, construct an angle of 60°.
11. Using ruler and compasses only, construct angles of
    (*a*) 30°, (*b*) 15°, (*c*) 120°, (*d*) 75°.

In each case make a separate diagram.

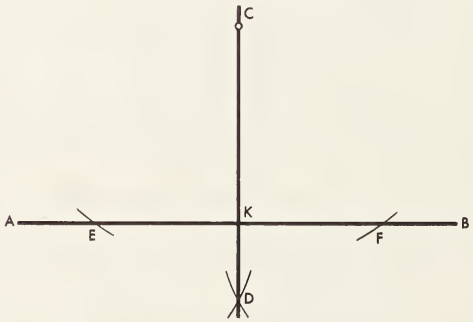
Distance from a Point to a Line

The distance from a point to a line is understood to be the distance measured along the perpendicular from the point to the line.

To Construct a Perpendicular from a Point to a Line

Let **C** be a given point and **AB** a given line not passing through **C**.

It is required to draw a line from **C** perpendicular to **AB**.



Method: With centre **C** and sufficient radius to cut **AB** draw arcs at **E** and **F** respectively.

With centre **E** and any convenient radius draw an arc on the opposite side of **AB**.

With centre **F** and the same radius draw an arc to intersect the former one at **D**.

Draw a line from **C** through **D** to intersect **AB** at **K**.

CK is perpendicular to **AB**.

Exercise

A

1. Explain the difference between a perpendicular to a line and the right bisector of the same line.

B

2. Draw a horizontal line and mark a point above it. From this point draw a line perpendicular to the given line. Describe your method.
3. Place a point below a horizontal line and from that point draw a perpendicular to the line. Describe your method.
4. (*a*) Draw a line making an angle of 45° with a horizontal line. Select a point on the horizontal line. From this point draw a perpendicular to the oblique line.
   (*b*) Repeat this problem using an oblique line drawn at an angle of 75° to the horizontal.
5. Construct a triangle with sides 3″, 2″ and $2\frac{1}{2}$″ respectively. Select one vertex and draw a line perpendicular to the opposite side. This perpendicular is called the *altitude* of the triangle. The side to which it is drawn is then considered the *base* of the triangle.
6. Construct a triangle **ABC** in which **AB** is 2″, **BC** is $3\frac{3}{8}$″ and **AC** is $2\frac{7}{8}$″. Using **BC** as base, construct and measure the altitude.
7. Draw an isosceles triangle **DEF** in which the lengths of **DE** and **DF** are equal. In an isosceles triangle the third side, not included in the equal sides, is usually considered the base. Draw the altitude of the triangle **DEF**. Notice the position of the point where it meets the base.
8. Draw a triangle **MNK** in which **MN** is $1\frac{1}{2}$″, **NK** is 2″ and **MK** is 3″. Consider **NK** the base and draw the altitude from the vertex **M** to **NK**. In this case it will be necessary to produce the base **KN**.

Parallel Lines

Observe the direction of the lines on the wall or the floor of your room. All the lines on the same *flat* surface are in the same *plane*. Find two straight lines on a plane which appear to run in exactly the same direction. If the plane with these lines were extended farther in each direction, do you think the lines which you selected would meet each other? If these lines would not meet when extended indefinitely in either direction they would be parallel lines.

Parallel lines are two or more straight lines, on the same plane, which do not meet when extended in either direction.

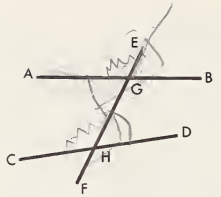


Diagram 1

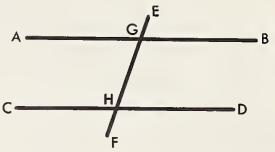


Diagram 2

In diagram 1, **AB** and **CD** are any two lines. Another line **EF** is drawn across **AB** and **CD**. **EF** intersects **AB** and **CD** at **G** and **H** respectively. **EF** is called a **transversal.**

The angles **AGH** and **GHD** are a pair of **alternate angles.** Name another pair of alternate angles in the diagram.

The angles **EGA** and **GHC** are a pair of **corresponding angles.** Name three other pairs of corresponding angles.

In diagram 2, **AB** and **CD** may be considered for practical purposes to be a pair of parallel lines. Measure and record the size of the alternate angles **AGH** and **GHD**. How do these angles compare when the lines are parallel? Compare the sizes of the other pairs of alternate angles.

Measure and compare the size of the angles in each pair of corresponding angles.

Using your ruler edges to give a pair of lines approximately parallel, draw any transversal and compare the sizes of (*i*) the alternate angles (*ii*) the corresponding angles.

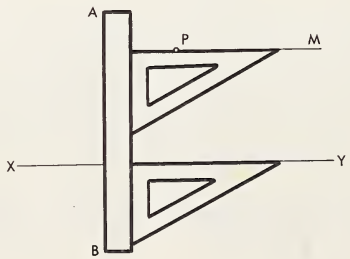
To Draw a Line Through a Point Parallel to a Given Line

Method 1:

Using a Set Square and Ruler draw a line through P parallel to XY.

Let **XY** be a given line and **P** a given point.

It is required to draw through **P** a line parallel to **XY**.



Method: Place the set square on the line **XY** as shown in the diagram. Place the ruler **AB** so that its edge is along the other side of the set square. Now, holding the ruler steady, move the set square along the ruler until the same edge is at the point **P**. Draw the line **PM**. **PM** is parallel to **XY**. By this method the ruler serves as the transversal and the set square makes the corresponding angles equal.

Therefore the lines are parallel.

Method 2:

Using Compasses and Ruler draw a line through any point P parallel to AB.

Let **AB** be a given line and **P** a given point.

It is required to draw a line through **P** parallel to **AB**.

Method: On the line **AB** mark any two points **K** and **R**.

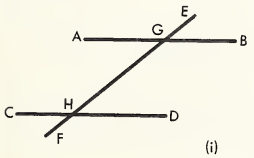
With centre **P** and radius **KR** draw an arc above **R**.

With centre **R** and radius **PK** draw an arc to intersect the first arc at **Q**. Join **PQ**. Draw a line through **RQ**.

PQ is parallel to **AB**.

Exercise

A



(i)

(ii)

1. In diagram (*i*) if **AB** is parallel to **CD** and $\angle$**EGB** = 40°, state the size of
   (*a*) $\angle$**AGH**, (*b*) $\angle$**GHD**, (*c*) $\angle$**CHF**, (*d*) $\angle$**EGA**,
   (*e*) $\angle$**CHG**, (*f*) $\angle$**FHD**, (*g*) $\angle$**HGB**.

2. Diagram (*ii*) illustrates a quadrilateral in which both pairs of the opposite sides are parallel. Such a quadrilateral is a **parallelogram**. If $\angle$**ABC** is 70°, what will be the size of each of the other angles?

B

3. Draw a horizontal line and mark a point above it. Through this point draw a line parallel to the given line. Describe your method.

4. Through a given point draw a line parallel to a given vertical line. Describe your method.

5. Draw a quadrilateral **ABCD** in which **AB** = 2″, **BC** = 3″, $\angle$**B** = 60° and **AD** is parallel to **BC** and **DC** is parallel to **AB**.
   (*a*) Name this quadrilateral.
   (*b*) Measure and compare the lengths of **AD** and **BC**; of **AB** and **CD**.

6. Construct an isosceles triangle **ABC** with **AB** = **AC**. Through **A**, draw a line **AD** parallel to **BC**. Measure and compare $\angle$**DAC** with $\angle$**C**.

7. Draw a quadrilateral **ABCD** in which **BC** is 3″, $\angle$**B** is 60°, **AB** is 1″, **AD** is parallel to **BC** and 2″ in length.

   A quadrilateral with two sides parallel is a **trapezoid.** This quadrilateral is often referred to as a **trapezium**. In this text it will be considered that a **trapezium** is a quadrilateral with no two sides parallel.

8. Construct a parallelogram **ABCD** with $\angle$**B** = 45°, **AB** = 2″ and **BC** = 4″. With ruler and compasses construct the perpendicular **AH** from **A** to **BC**. Take any other point **K** in **AD** and construct **KT** perpendicular to **BC**. Compare the length of **AH** and **KT**.

   The **altitude of a parallelogram** is the *distance* from any point on one side to the opposite side. The side to which the altitude is drawn is the **base** of the parallelogram.

9. With ruler and compasses only, construct the parallelogram **ABCD** with $\angle$**B** = $67\frac{1}{2}$°, **AB** = 6·3 cm. and **BC** = 4·7 cm. Construct and measure the altitude drawn to the base **BC**.

C

10. Using the set square method, draw a line through a point **A** parallel to a line **BC**.

11. (*a*) Draw any triangle **ABC**. Bisect the side **AB** at **D**. Through **D** draw a line parallel to **BC** to intersect **AC** at **E**.
    (*b*) Compare the length of **AE** and **EC**.
    (*c*) Compare the length of **DE** and **BC**.

12. Construct a line parallel to and 2″ distant from a given straight line.

Review Exercise

A

1. State a definition for
   (*a*) triangle, (*b*) isosceles triangle, (*c*) scalene triangle,
   (*d*) acute-angled triangle.

2. If the acute angle at the vertex of an isosceles triangle is 70°, what is the size of the other two angles?
3. What are the sizes of the angles in a right-angled isosceles triangle?
4. If the three angles of a triangle are equal, what is the size of each angle?
5. If two angles of a triangle are 40° and 60°, what is the size of the third angle?
6. What is the difference between 90° and 85° 10′?
7. What sets of dimensions determine definitely the shape of a triangle? What three dimensions will not determine a triangle uniquely?
8. Define
   (*a*) bisect, (*b*) right bisector, (*c*) perpendicular.
9. Define
   (*a*) parallel lines, (*b*) vertically opposite angles,
   (*c*) alternate angles, (*d*) transversal.

B

10. Construct triangle **ABC** in which **AB** $= 3\frac{1}{4}''$, **BC** $= 3''$ and **CA** $= 2\frac{3}{4}''$. Measure $\angle$**BAC**.
11. Construct triangle **DEF** with $\angle$**EDF** $= 65°$, **ED** $= 4{\cdot}3$ cm., **DF** $= 5{\cdot}5$ cm. Bisect $\angle$**EDF** and let the bisector cut **EF** at **G**. Measure the length of **EG**.
12. Construct triangle **XYZ** with **YZ** $= 3''$, $\angle$**Y** $= 55°$, $\angle$**Z** $= 40°$. Construct the altitude from **X** to **YZ** and find its length.
13. Construct an equilateral triangle of any size and circumscribe a circle round it.
14. Construct a large scalene acute-angled triangle. Bisect two angles and produce the bisectors to meet at **O**. Construct **OH** a perpendicular from **O** to one side with **H** as its base. Inscribe a circle in the triangle.
15. Construct an isosceles triangle **ABC** with equal sides **AB** and **AC**. Choose any point **P** in **AB** and through **P** construct a line parallel to **BC** to meet **AC** at **Q**. What kind of a figure is **PQCB**?
16. Construct a parallelogram **ABCD**. Bisect **AC** at **E**. Through **E** draw a line parallel to **BC** to meet **AB** at **M** and **CD** at **N**. What kind of a figure is **MBCN**?
17. Construct an angle of 45° and one of 60° without using a protractor.
18. Construct angles of 30°, 135° and 75° without using a protractor.

CHAPTER 3

Integral Measures

Your success in solving problems will increase as you learn to recognize and use a systematic method. As a guide, the following is suggested.

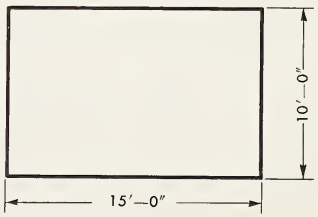
1. Diagrams should be neatly drawn and carefully dimensioned.
2. Relationships that exist between measures must be understood.
3. Clear statements should be written to indicate:
   (a) the **quantity,**
   (b) the **unit of measurement,**
   (c) the required **measure** expressed in terms of the known measures.
4. An **orderly method** should be used in simplifying the expressed measures.

The **quantities** that we meet in everyday life include those found in the measurement of distance, area, volume, weight, time, rotation and money value. You are already familiar with the units of measurement in the English System. The greater part of the people of the world, however, use the metric units of measurement. It will be necessary to learn to use both systems.

Let us study a simple problem to see how we would arrange a solution.

Example 1: The length of a rectangle is 15 feet and its width is 10 feet. Calculate: (*a*) its perimeter, (*b*) its area.

Whenever possible make a careful diagram to represent your problem.



Solution:

The length of the rectangle in feet is 15.
The width of the rectangle in feet is 10.

(*a*) The perimeter of the rectangle in feet is $2(10 + 15)$
$= 2 \times 25$
$= 50$

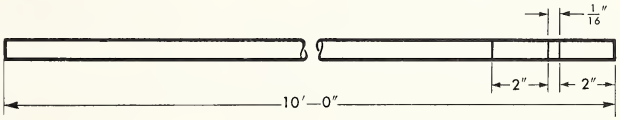
(*b*) The area of the rectangle in square feet is 10×15
$= 150.$

Each statement indicates clearly
—the quantity,
—the unit of measurement,
—the numerical measure.
By the application of a known relationship, the "new" measure is "expressed" in terms of the known measures.
An orderly method is used to simplify the expressed measures.

Example 2: For a certain job, steel pins 2 inches long are required.

(*a*) How many of these pins can be cut from a piece of stock 10 feet long if $\frac{1}{16}$ in. is wasted in each cut?

(*b*) What length of stock is left after the pins have been cut off?



Solution:

(*a*) The length of the stock in inches is 10×12
$= 120.$

The length of the stock required for each pin in inches is
$2 + \frac{1}{16}$
$= 2\frac{1}{16}.$

The number of pieces is $120 \div 2\frac{1}{16}$
$= 120 \div \frac{33}{16}$
$= \overset{40}{\cancel{120}} \times \frac{16}{\cancel{33}_{11}}$
$= \frac{640}{11}$
$= 58\frac{2}{11}.$

Therefore, the number of pins is 58.

(*b*) The length of the remaining stock in inches is $\frac{2}{11}$ of $2\frac{1}{16}$
$= \frac{\overset{1}{\cancel{2}}}{\cancel{11}_{1}} \times \frac{\overset{3}{\cancel{33}}}{\cancel{16}_{8}}$
$= \frac{3}{8}.$

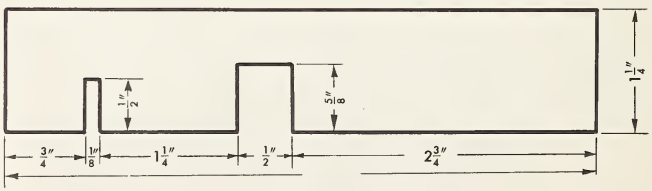
Exercise

A

1. State at least five quantities that are distances.
2. What quantity is measured when the following units are used:
   (*a*) pounds, (*b*) seconds, (*c*) miles per hour,
   (*d*) dollars, (*e*) centimetres, (*f*) degrees,
   (*g*) square feet, (*h*) cubic centimetres, (*i*) watts?
3. What relationship is used to calculate
   (*a*) the area of a rectangle,
   (*b*) the perimeter of a rectangle,
   (*c*) the third angle of a triangle when two are known,
   (*d*) the perimeter of a triangle,
   (*e*) the length of a desk in inches when it is stated in centimetres?

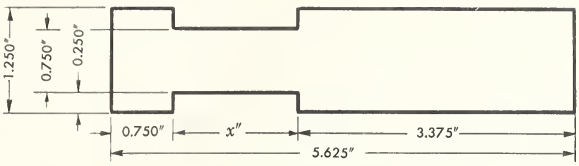
B

4. How much greater is the area of a rectangle 15 feet long and 12 feet wide than one 3 yards long and 2 yards wide?
5. A man has two lots each containing 6,000 sq. ft. One lot is 50 ft. wide, the other 75 ft. wide. What will be the difference, if any, in the length of the fences required to enclose each of these lots?
6. A job of building and installing cupboards in a new house required carpenters' time as follows: 2 men for 4 hours each, 3 men for 2 hours each, 5 men for 8 hours each. Find the total number of hours of labour required for the job.
7. An order for coal was delivered in four truck loads the weights of which were 8,125 lb., 7,865 lb., 8,275 lb., and 8,344 lb. respectively.
   (*a*) What was the average weight per load?
   (*b*) What was the cost if the price of this coal was $15 per ton?
8. (*a*) What is the total length of the piece of material illustrated?



(*b*) Make a neat copy of this diagram in your notebook and write in the dimensions, using decimal fractions instead of common fractions. Express the over-all length in decimal notation.

9. (*a*) Find the missing dimension indicated by x'' in this diagram.



(*b*) Make a neat copy of this diagram, using common fractions to indicate the dimensions.

10. A rectangle is $\frac{7}{8}$ of an inch long and $\frac{3}{4}$ of an inch wide.
   (*a*) What is its perimeter?
   (*b*) What is its area?
   (*c*) What will be the area of $\frac{2}{3}$ of this rectangle?

11. The length of a rectangle is $3\frac{1}{4}$ ft. and its width is $2\frac{1}{2}$ ft.
   (*a*) What is its perimeter?
   (*b*) What is its area?

12. The area of a rectangular opening for a ventilator is to be 6 sq. ft. What would be the corresponding length for each of the proposed widths: (*a*) $1\frac{1}{2}$ ft., (*b*) $\frac{1}{2}$ ft., (*c*) $\frac{3}{4}$ ft., (*d*) $1\frac{1}{4}$ ft.?

C

13. A packing crate, in the form of a rectangular box 3 ft. deep, 4 ft. wide and 5 ft. long, is to be reinforced by fastening metal stripping on all of its edges. Allowing 1 ft. extra for waste in cutting, find the cost of this material at 10 cents a foot.

14. In travelling from Toronto to Kingston, a total distance of 160 miles, a driver makes an average speed of 45 miles per hour on the open road and averages 20 miles per hour in restricted areas. It is estimated that 25 miles of the trip is made at the slower rate. How much time should be allowed for the entire trip? What was the average speed for the entire trip?

15. A contractor is asked by a customer to quote a price on a small job. The contractor estimates 15 hours of labour at $2·25 per hour, about 300 feet of Romex cable at 9 cents a foot, $10 for small articles and $15 for overhead expenses. What quotation will he give the customer?

Order of Operations

In the previous exercise we have seen that by the proper use of arithmetical signs of operation the required measure may be expressed in terms of the known measures. Here are some examples which illustrate the order which must be followed in simplifying such expressions.

Example 1: Calculate the perimeter of a rectangle 4 ft. long and 3 ft. wide.

Solution: Perimeter in feet is $2(4 + 3)$

$$= 2 \times 7$$
$$= 14.$$

Example 2: A boy purchased one pencil costing 10 cents and 2 erasers costing 4 cents each. What was the total cost?

Solution: Total cost in cents is $10 + 2 \times 4$

$$= 10 + 8$$
$$= 18.$$

Example 3: A dealer delivered 3 tons of coal for \$54 and later in the season charged \$80 for 4 tons of a similar grade of coal. How much per ton has the price increased?

Solution: The increase in price per ton in dollars is $80 \div 4 - 54 \div 3$

$$= 20 - 18$$
$$= 2.$$

Example 4: In the first week of July, a boy earned \$25 and spent \$7; in the second week he spent \$3 and in the third week he earned \$8. What was his balance at the end of the time?

Solution:

The balance in dollars is $25 - 7 - 3 + 8$ or $(25 + 8) - (7 + 3)$

$$= 18 - 3 + 8 \qquad = 33 - 10$$
$$= 15 + 8 \qquad = 23.$$
$$= 23.$$

Example 5: Find the length of the rectangle the width of which is 5 ft. and the area of which is equal to that of a rectangle 8 ft. by 10 ft.

Solution: The length of the rectangle in feet is $8 \times 10 \div 5$

$$= 80 \div 5$$
$$= 16.$$

Example 6: If 3 oranges are known to cost 15 cents what will be the cost of 4 oranges?

Solution:

The cost of 4 oranges in cents is $15 \div 3 \times 4$

$$= 5 \times 4$$
$$= 20.$$

Example 7: Simplify $\frac{2}{3}(6 + 9) + 8 \div 4 \times 3 - 45 \div \frac{3}{5}$ of 25.

Solution: $\frac{2}{3}(6 + 9) + 8 \div 4 \times 3 - 45 \div \frac{3}{5}$ of 25

$$= \tfrac{2}{3} \times 15 + 2 \times 3 - 45 \div 15$$
$$= 10 + 6 - 3$$
$$= 13.$$

Summary. The following rules should be observed when simplifying expressions involving combined operations:

(1) **Simplify the parts within the brackets first.**

(2) **If an operation involving "of" signs occurs, simplify this next.**

(3) **Perform the operations of multiplication and division in the order in which they occur.**

(4) **Combine the resulting terms using addition or subtraction as indicated.**

Exercise

A

Simplify:

1. $2(8 + 3)$.

2. $5 \times 3 - 2 \times 3$.

3. $\frac{2}{3}$ of $6 + 8$.

4. $\frac{3}{5}$ of $(7 + 3)$.

5. $6 \times 4 \div 3$.

6. $6 \div 3 \times 4$.

7. $5 - 2 + 7 - 3 + 2$.

8. $8 - 3 \times 2$.

9. $\dfrac{5 + 7}{3}$.

10. $21 - 2 \times 0{\cdot}1$.

11. $\dfrac{71 + 37}{2 \times 9}$.

12. $15 - \dfrac{12}{3}$.

13. $\dfrac{15 - 12}{3}$.

14. $\frac{1}{3}(15 - 12)$.

B

15. Express, in terms of the measures given, the perimeter of a rectangle which is 8 feet long and 5 feet wide. Simplify this expression.

16. Express, in terms of the measures given, the difference in area of two squares one of which is 8 feet long and the other 7 feet long. Simplify the expression obtained.

17. Express, in the most convenient form, the difference in the area of a rectangle 8 feet by 6 feet and a rectangle 12 feet by 3 feet.

18. Express, in terms of the measures given, the average of these three measurements 36 in., 37 in. and 38 in.

Simplify:

19. $3(5 + 7) - 2(6 - 3)$.

20. $2{\cdot}5 \times 3 - 0{\cdot}5 \times 4$.

21. $3{\cdot}2 - 1{\cdot}5 + 0{\cdot}3$.

22. $3(4 + 8) + 2(7 - 4)$.

23. $45 \div \frac{3}{5}$ of 25.

24. $45 \div \frac{3}{5} \times 25$.

25. $\dfrac{4 - (8 - 5)}{5} + \dfrac{8 + (6 - 4 + 3)}{3}$.

26. $\dfrac{(9 - 4 + 3) - (8 - 2 - 4)}{15}$.

27. $\frac{3}{5}$ of $35 \div \frac{1}{4}$ of 28.

28. $\dfrac{3 + 7 - 4}{2} + \frac{3}{4}$ of $76 - 21$.

C

29. $\frac{8{\cdot}5 - 0{\cdot}9 + 0{\cdot}5}{9} + 4{\cdot}02 - 3{\cdot}7 + \frac{(6{\cdot}15 - 4{\cdot}26)}{3}$.

30. $0{\cdot}64 + 5 \times 0{\cdot}127 - \frac{3}{4}$ of $1{\cdot}64 + 0{\cdot}0015 \div \frac{2}{3}$ of 15.

31. $\frac{12{\cdot}8 \div 4}{8} \div \frac{2}{5}$ of $20 + 0{\cdot}125 \times 3$.

Types of Numbers

You are familiar with the various types of numbers such as **integers** sometimes called whole numbers— **common fractions** and **decimal fractions**.

A **mixed number** consists of an integer and a fraction. It may be expressed as an integer and a common fraction such as $1\frac{1}{2}$ or as an integer and a decimal fraction such as 2·5.

An **expression** is a combination of numbers connected by the signs of operation.

Example (*a*) $186 + 32 - 4$.

(*b*) $2 \times 3 + 8 \times 5$.

The **terms of an expression** are the numbers which are added or subtracted. Expression (*a*) above has 3 terms; expression (*b*) has two terms.

Factors are numbers which are multiplied to form a product.

Exercise

A

1. State the meanings of the terms listed below as they apply to mathematics:

(*a*) addend, (*b*) sum, (*c*) subtrahend,
(*d*) minuend, (*e*) difference, (*f*) product,
(*g*) factor, (*h*) quotient, (*i*) dividend,
(*j*) divisor, (*k*) remainder.

2. What must be added to 387 to make it equal to 500?

3. What is the difference when the subtrahend is 8,756 and the minuend 10,000?

4. Express as an integer the sum when 5 is taken as an addend 3 times.

5. Express as an integer the product when 5 is taken as a factor 3 times.

6. Simplify:

(*a*) $27 - 3 \times 8$, (*b*) $27 \div 3 \times 9$, (*c*) $81 - 8 \times 9$,
(*d*) $(90 - 10) \times 3$, (*e*) $28 - 32 + 7$, (*f*) $2 \times 4 + 5 \times 6$.

B

7. Find the sum when the following numbers are used as addends: 25, 3, 275, 453, 2,007.

8. Write an expression to show that the sum of 5 and 6 is to be subtracted from the difference between 25 and 13. Simplify the expression.

9. The block of an automobile engine weighs 409 lb. Find the total weight of a shipment containing 765 blocks.

10. What is the total power, measured in watts, being used in a small office building where there are 19 40-watt bulbs, 13 60-watt bulbs and 28 25-watt bulbs? Assume that all the bulbs are operating at the same time.

11. (*a*) How many pieces 7″ in length can be cut from a rod 96″ in length?
(*b*) What will be the length of the piece remaining?

12. A piece of cloth 72 inches long is divided into 5 equal pieces. How long is each piece?

13. Calculate the area of a square the length of each side of which is 17 feet.

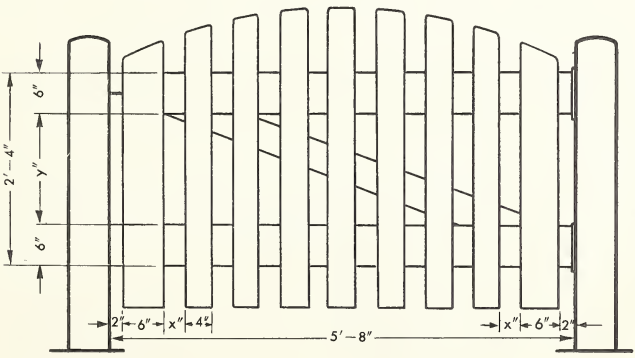
14. What will be the volume of a cube each side of which measures 11 inches?

15. A number of parts for a lathe were shipped to your school. The weights were: base 492 lb., headstock 73 lb., face plate 5 lb., bed 319 lb., tailstock 57 lb., pulleys 43 lb., and motor 119 lb. What was the total weight?

16. An electrician cuts off the following lengths of wire: 28′, 48′ 6″, 115′, 29′ 6″, 48′ 4″, 34′ 5″ and 49′ 9″.
(*a*) How much wire was used?
(*b*) How much is left if the roll originally contained 500′?

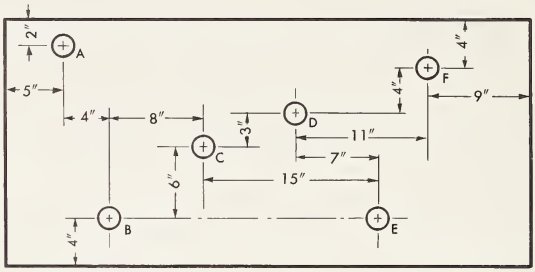
17. A piece of strap iron 1″ by $\frac{1}{8}$″ is 12′ long. A boy is asked to cut off the following lengths: 9″, 15″, 18″, 13″, $14\frac{1}{2}$″, $18\frac{1}{2}$″, $23\frac{1}{2}$″ and $12\frac{1}{2}$″. What length is left?

18. A garden gate is shown in the accompanying diagram. Find:
(*a*) the length *y*″ between the two horizontal supports;
(*b*) the width *x*″ of each of the equal spaces between the pickets if each picket is 4″ wide.



19. A certain bar of steel is 7 feet long. It is to have 10 equally spaced holes drilled in it but the end holes must be centred exactly 10″ from the end of the bar. What is the distance between the centres of consecutive holes? It is suggested that you work from a neat diagram.

20.



The figure above is the lay-out for drilling a panel board for the electrical laboratory. Find the following dimensions:

(*a*) the horizontal distance from **B** to **E**,
(*b*) the horizontal distance from **A** to **F**,
(*c*) the vertical distance from **F** to **E**,
(*d*) the vertical distance from **A** to **C**,
(*e*) the length and width of the board.

21. If the panel board as shown in Problem 20 cost $34·00, find the cost of equipping 3 laboratories if each laboratory requires 5 panels.

Powers of Numbers

Powers are products in which all the factors are identical.

The measure of the area of a square which is 17 feet in length is expressed by 17×17. This may be written 17^2 and is read 17 squared or 17 to the second.

The measure of the volume of a cube which is 11 inches to the side could be expressed as 11^3 which means $11 \times 11 \times 11$ and is read 11 cubed or 11 to the third.

In the same manner $5^4 = 5 \times 5 \times 5 \times 5$.

The number used repeatedly as a factor in a power is called the **base of the power** and the small number at the upper right of the base which indicates the number of equal factors is called the **index**.

Example: Find the value of $5^3 - 4^3$.

Solution: $5^3 - 4^3 = 5 \times 5 \times 5 - 4 \times 4 \times 4$
$= 125 - 64$
$= 61.$

Exercise

A

1. Express as a power, the measure of the area of a square when the length of its side is 7 feet.
2. Express as a power, the measure of the volume of a 15-foot cube.
3. Find the value of each of the following as integers:

   (*a*) 5^2, (*b*) 8^2, (*c*) 2^3,
   (*d*) 4^2, (*e*) 10^3, (*f*) 7^2,
   (*g*) 1^3, (*h*) 3^4, (*i*) 2^5,
   (*j*) 2^7, (*k*) 20^2, (*l*) 7^3,
   (*m*) $3^3 \times 5^3$, (*n*) $2^3 \times 3^2$, (*o*) $1^5 \times 12^2$.

4. Express:

   (*a*) 25 as a power of 5, (*b*) 36 as a power of 6,
   (*c*) 1,728 as a power of 12, (*d*) 16 as a power of 2,
   (*e*) 1,000,000 as a power of 10.

5. Choosing suitable bases, express as powers in at least two different ways:

   (*a*) 16, (*b*) 64,
   (*c*) 81, (*d*) 625,
   (*e*) 10,000.

6. Simplify:

   (*a*) $3^2 - 2^2$, (*b*) $10^2 \div 5^2$, (*c*) $3^2 + 4^2$,
   (*d*) $(2 + 3)^2$, (*e*) $10^3 \div 5^3$, (*f*) $(8 - 7)^3$.

B

7. Make a table showing the numerical value of the second power of each of the first 15 integers.
8. Make a table showing the numerical value of the third power of each of the first 10 integers.
9. Express in simplified form:

   (*a*) $7 - 2 \times 3$, (*b*) $6(2 + 7)$,
   (*c*) $7^2 - 4^2$, (*d*) $156 - \dfrac{25 \times 24}{8}$,
   (*e*) $3^2(5^2 - 4^2)$, (*f*) $3(5^3 - 4^2) + 2(8^2 - 3^3)$,
   (*g*) $\dfrac{15(27 - 9)}{6}$.

10. Find the value of:

    (*a*) $15^2 + 25^2$, (*b*) $42^2 - 38^2$,
    (*c*) $64^2 + 61^2$, (*d*) $48^2 - 47^2$,
    (*e*) $15^2 + 8^2$, (*f*) $98^2 - 97^2$.

Prime Factors

A heating expert has found that a certain room requires a rectangular heating duct the area of which is to be 36 sq. in. What are the possible dimensions of this rectangle?

For the present we shall consider integral measures only. If fractional measures were included, there would be no limit to the number of possible sets of dimensions.

Length of one side of duct in inches	1	2	3	4	6	9	12	18	36
Length of other side of duct in inches	36	18	12	9	6	4	3	2	1

This table gives several possible dimensions for the heating duct. When 36 is divided by any one of the numbers 1, 2, 3, 4, 6, 9, 12, 18 the quotient is an integer. Each of these numbers is a **factor** of 36.

Two of the factors of 36, namely 2 and 3, have no integral factors except themselves and 1. These are known as **prime numbers**. **Prime numbers** have no integral factors except themselves and one.

Numbers such as 4, 6, 9, 12 can be expressed as the product of two or more integers other than 1. Such numbers are **composite numbers**.

Composite numbers are numbers which can be expressed as the product of two or more integral factors. Composite numbers can be factored by using prime numbers.

Hence, 36 as the product of its prime factors is $2 \times 2 \times 3 \times 3$.

Example 1: Find the prime factors of 72.

Solution: $72 = 8 \times 9$ Eight and nine are composite factors of 72.

$= 2 \times 2 \times 2 \times 3 \times 3$ These are the prime factors of 72.

Example 2: Express 6,930 as the product of its prime factors.

Solution: Use prime numbers successively as divisors in this manner:

$$\begin{array}{r|l} 2 & 6930 \\ 3 & 3465 \\ 5 & 1155 \\ 3 & 231 \\ 7 & 77 \\ & 11 \end{array}$$

$$\therefore 6930 = 2 \times 3 \times 3 \times 5 \times 7 \times 11.$$

Checks for Divisibility

When factoring numbers you should use, wherever possible, a quick check for divisibility. Some simple checks are:

(1) If the last digit of a number is 0, 2, 4, 6, 8, the number is divisible by 2.

(2) If the sum of the digits of a number is divisible by 3, the number is divisible by 3.

(3) If the last digit of a number is 0 or 5, the number is divisible by 5.

Exercise

A

1. State the first 10 prime numbers.
2. State the first 10 composite numbers.
3. State a prime number that is even.
4. State some composite numbers that are odd.
5. Give the prime factors of
   (*a*) 9, 12, 30, 25, 27; (*b*) 42, 38, 56, 81, 100.
6. Which of the first 9 integers are factors of 204?
7. State all the numbers which are factors of 12, 15, 42 and 48.

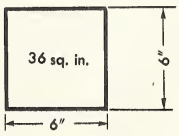
B

8. Express each of the following numbers as the product of its prime factors:
   (*a*) 76, (*b*) 54, (*c*) 91, (*d*) 343,
   (*e*) 48, (*f*) 110, (*g*) 144, (*h*) 512.
9. Find the prime factors of
   (*a*) 2,025, (*b*) 5,625, (*c*) 441.
10. Using the method of successive division by prime numbers express each of the following numbers as the product of its prime factors:
    (*a*) 2,310, (*b*) 1,260, (*c*) 585, (*d*) 9,282, (*e*) 1,296.
11. A rectangle is to have an area of 225 sq. in.
    (*a*) Find four sets of integral dimensions for this rectangle exclusive of 1″ and 225″.
    (*b*) Find the perimeter of each of these rectangles.
    (*c*) Which of these has the least perimeter?

Square Root

Examine again the table giving the sets of dimensions for the rectangle the area of which is 36 sq. in. In one set the two dimensions are equal.

This, of course, would be the dimensions of a special type of rectangle called a **square**. This relationship expresses 36 as a product of its two equal factors 6×6. Whenever a number can be expressed as the product of two equal factors, **one** of these factors is called its square root.



The square root of a number is one of its two equal factors. The symbol $\sqrt{\ }$ placed over a number indicates that the square root is to be taken.

Since $9 = 3 \times 3$ $\quad\quad$ Since $49 = 7 \times 7$

$\therefore \sqrt{9} = 3.$ $\quad\quad$ $\therefore \sqrt{49} = 7.$

Example 1: Find the square root of 1,764.

Solution: Use the method of division to find the prime factors of 1,764.

$$1764 = 2 \times 2 \times 3 \times 3 \times 7 \times 7$$

Arrange the factors in *two* equal groups:

$$1764 = 2 \times 3 \times 7 \times 2 \times 3 \times 7$$
$$= 42 \times 42$$

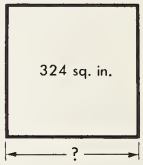
Since $1764 = 42 \times 42$; $\therefore \sqrt{1764} = 42$.

$$\begin{array}{r|l} 2 & 1764 \\ 2 & 882 \\ 3 & 441 \\ 3 & 147 \\ 7 & 49 \\ & 7 \end{array}$$

Example 2: Calculate the length of one side of a square the area of which is 324 sq. in.

Solution: The length of one side of the square in inches is $\sqrt{324}$

$$= \sqrt{2 \times 3 \times 3 \times 2 \times 3 \times 3}$$
$$= \sqrt{18 \times 18}$$
$$= 18.$$



$$\begin{array}{r|l} 2 & 324 \\ 2 & 162 \\ 3 & 81 \\ 3 & 27 \\ 3 & 9 \\ & 3 \end{array}$$

Example 3: Seven rectangular heating ducts each 8 inches by 14 inches are to be supplied by one square duct. What should be the dimensions of this square duct so that its area will be equal to the combined area of the 7 rectangular ones?

Solution:

The area of the 7 rectangular ducts in sq. in. is $7 \times 8 \times 14$.
The length of one side of the square duct in inches is

$$\sqrt{7 \times 8 \times 14}$$
$$= \sqrt{7 \times 2 \times 2 \times 2 \times 2 \times 7}$$
$$= \sqrt{7 \times 2 \times 2 \times 7 \times 2 \times 2}$$
$$= \sqrt{28 \times 28}$$
$$= 28.$$

Exercise

A

1. State the value of

(*a*) 5^2, (*b*) 7^2, (*c*) 9^2, (*d*) 12^2, (*e*) 15^2,
(*f*) 13^2, (*g*) 20^2, (*h*) 16^2, (*i*) 17^2, (*j*) 14^2.

2. Find the value of
   (*a*) $\sqrt{9}$, (*b*) $\sqrt{25}$, (*c*) $\sqrt{81}$, (*d*) $\sqrt{144}$,
   (*e*) $\sqrt{169}$, (*f*) $\sqrt{196}$, (*g*) $\sqrt{400}$, (*h*) $\sqrt{1}$.
3. If the area of a square is 49 sq. in., what is the length of one side?
4. Simplify:
   (*a*) $\sqrt{9 \times 25}$, (*b*) $\sqrt{2 \times 2 \times 2 \times 2}$,
   (*c*) $\sqrt{3 \times 5 \times 7 \times 5 \times 7 \times 3}$,
   (*d*) $\sqrt{27 \times 3}$, (*e*) $\sqrt{4 \times 7 \times 5 \times 5 \times 7}$.

B

5. Find the length of one side of a square the area of which is
   (*a*) 121 sq. in., (*b*) 196 sq. in.
6. State the square root of each of the following:
   (*a*) 8,100, (*b*) 14,400, (*c*) 1,600, (*d*) 225,
   (*e*) 625, (*f*) 400, (*g*) 324, (*h*) 2·25.
7. By first expressing the number as the product of its prime factors, calculate the square root of each of the following:
   (*a*) 2,025, (*b*) 1,225, (*c*) 441, (*d*) 2,601,
   (*e*) 3,136, (*f*) 7,744, (*g*) 1,521, (*h*) 11,025.
8. Find the length of one side of a square the area of which is 3,969 sq. ft.
9. Calculate the length of the side of a square equal in area to the sum of the areas of two squares—the side of one of which measures 21 inches and the side of the other, 28 inches.
10. What would be the length of a side of a square equal in area to the difference of two squares—the side of one of which measures 65 feet and the side of the other, 25 feet?
11. Simplify:
    (*a*) $\sqrt{3^2 + 4^2}$, (*b*) $\sqrt{10^2 - 8^2}$, (*c*) $\sqrt{9^2 + 12^2}$,
    (*d*) $\sqrt{25^2 - 24^2}$, (*e*) $\sqrt{41^2 - 40^2}$, (*f*) $\sqrt{61^2 - 60^2}$.
12. What would be the difference between the length of a fence required to enclose a 10-acre field in the form of a rectangle 20 rods wide and one in the form of a square?
    (Note: 1 acre is 160 sq. rd.)

Multiples

A gardener requires a number of stakes 3 feet in length. He has discovered that it is more economical to buy the lumber and cut it into the required lengths than to order the stakes ready made. Suggest some suitable lengths of material that he might consider so that he would have no waste when he cut it into 3-foot lengths.

It is evident that the lengths in feet could be any one of 3, 6, 9, 12, 15, 18, etc. To obtain any one of these suggested measures we have multiplied 3 by an appropriate integer. The numbers 3, 6, 9, 12, 15, 51, 285, etc. are **multiples** of 3.

A multiple of a number is any product obtained by multiplying that number by an integer.

Lowest Common Multiple

Examine the following situation: A baking firm plans to make rectangular-shaped biscuits 2 inches by 3 inches. Suggest sizes for suitable large square pans in which biscuits of this size could be baked.

In order that the 2″ side could be accommodated, the measure of the side of the pan in inches could be any multiple of 2. To accommodate the 3″ side the measure of the pan in inches could be any multiple of 3.

Multiples of 2		2		4		6		8		10		12		14		16		18	etc.
Multiples of 3			3			6			9			12			15			18	

∴ a suitable size of square pan could be any one of 6″, 12″, 18″, etc.

Each of the measures 6, 12, 18, etc. is a common multiple of 2 and 3. The lowest common multiple of 2 and 3 is 6.

The **Lowest Common Multiple** of two or more numbers is the smallest number which will contain each of them as a factor.

Of course if the situation required finding a common multiple of two measures such as 60 and 72, the method used in the last example would be cumbersome. Since a multiple of a number contains that number as a factor, it must contain the product of all the prime factors of that number. The multiples of 60 contain the product of all the prime factors of 60. The multiples of 72 contain the product of all the prime factors of 72. Each of the common multiples of 60 and 72 must contain the product of all the prime factors of 60 and any additional prime factors necessary to contain the prime factors of 72.

∴ by factoring, $60 = \textcircled{2} \times \textcircled{2} \times \textcircled{3} \times \textcircled{5}$.

$72 = 2 \times 2 \times \boxed{2} \times 3 \times \boxed{3}$.

∴ the lowest common multiple of 60 and 72 is

$\textcircled{2} \times \textcircled{2} \times \textcircled{3} \times \textcircled{5} \times \boxed{2} \times \boxed{3}$.

L.C.M. of 60 and 72 is 360.

Exercise

A

1. State the L.C.M. of

(*a*) 2, 4, 6; (*b*) 3, 4, 6; (*c*) 4, 5, 6;

(*d*) 12, 18, 24; (*e*) 9, 12, 16; (*f*) 6, 10, 15.

2. State the prime factors of 20, 35, 36, 28, 18, 44.

3. Simplify:

(*a*) $4 + 5 \times 6$; (*b*) $\sqrt{16} + \sqrt{25} - \sqrt{49}$;
(*c*) $3^2(5^2 - 4^2)$; (*d*) $10^2 \div 5^2$.

4. Define: composite number, factor, square root, multiple.

5. Select the numbers which are common multiples of 4 and 6 from the following: 12, 16, 18, 24, 28, 30, 36, 40, 42, 48.

B

6. Find the L.C.M. of each of the following sets of numbers:

(*a*) 36, 48; (*b*) 16, 56, 21;
(*c*) 35, 65, 91; (*d*) 15, 34, 51, 85;
(*e*) 30, 38, 95; (*f*) 66, 132, 24;
(*g*) 63, 72, 126; (*h*) 92, 115, 120.

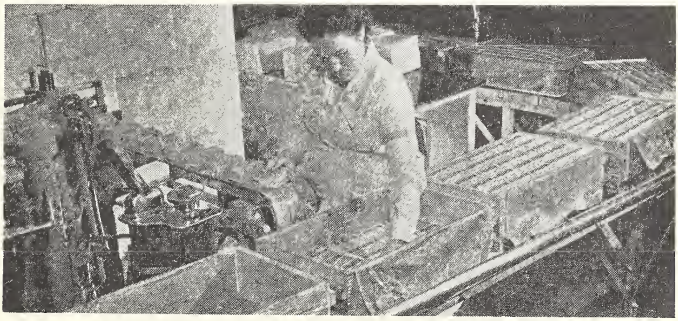
7. Find by factoring the L.C.M. of:

(*a*) 230 and 506; (*b*) 210, 462 and 546;
(*c*) 495, 891 and 1,155; (*d*) 481 and 1,665.

8. A manufacturer wishes to buy steel bars of such length that he can cut each bar into 6″ pieces, or 8″ pieces, or 10″ pieces without waste. What length of bar should he purchase?

C

9. A firm markets a product in a rectangular-shaped package 6″ long, 3″ wide and 2″ high. Submit the dimensions of 5 suitable packing cases in which at least 1 dozen of these might be shipped. Along with the dimensions of each carton indicate the number of packages it will hold. It is suggested that a summary of your results be arranged in tabulated form.



Courtesy National Film Board

Packing Butter

Suitable packing cases have been designed for this product.

10. If the shipping manager of the firm mentioned in Problem 9 requires that the cartons be cubical in shape, submit the dimensions of two such cartons with the number of packages each will contain.

11. Tubes of a certain kind of tooth paste are put in packages 6″ by 2″ by 1″. The manufacturer wishes to ship these in cartons each holding $\frac{1}{2}$ gross (72). Find the dimensions of at least 3 suitable cartons to hold this number of packages. Construct a table to show your results and draw a dimensioned diagram of the carton most suitable for shipping purposes.

Review Exercise

A

1. When the following units of measurement are used, what quantities are being discussed:

(*a*) acres, (*b*) cubic centimetres, (*c*) miles per hour,
(*d*) r.p.m., (*e*) hours, (*f*) degrees Fahrenheit,
(*g*) feet per second, (*h*) mills, (*i*) board feet,
(*j*) grams, (*k*) gallons?

2. Simplify:

(*a*) $6 + 4 \times 3$, (*b*) $24 \div 3 \times 8$, (*c*) $\frac{1}{2}$ of $16 + 4$,
(*d*) $3(8 - 5)$, (*e*) $(7 + 2)^2$, (*f*) $\sqrt{3^2 + 4^2}$.

3. State the results of the following operations:

(*a*) 32×100, (*b*) $1764 \div 100$, (*c*) $36912 \div 3$,
(*d*) $9801 \div 9$, (*e*) 2432×25, (*f*) $768945 \div 5$.

4. State the prime factors of

(*a*) 88, (*b*) 76, (*c*) 69, (*d*) 51, (*e*) 77.

5. Define: integer, prime number, composite number, square root, lowest common multiple.

6. State the value of:

(*a*) $\frac{258}{3}$, (*b*) $\frac{434}{7}$, (*c*) $\frac{3256}{8}$, (*d*) $\frac{594}{9}$,

(*e*) $\frac{90072}{6}$, (*f*) $\frac{492}{12}$, (*g*) $\frac{345}{15}$.

7. Distinguish between dividend and quotient.

8. State the value of 11^2, 7^3, 8^2, 8^3, 12^3, 3^4, 5^3, 2^5.

9. Express

(*a*) 36 as a power of 6, (*b*) 1000 as a power of 10,
(*c*) 64 as a power of 4, (*d*) 64 as a power of 8,
(*e*) 121 as a power of 11, (*f*) 125 as a power of 5,
(*g*) 81 as a power of 9, (*h*) 81 as a power of 3.

10. Find the square root of:

(*a*) 169, (*b*) 225, (*c*) 144, (*d*) 0·25, (*e*) 0·36, (*f*) $\frac{4}{9}$,
(*g*) $\frac{16}{25}$, (*h*) $\frac{81}{100}$, (*i*) $6\frac{1}{4}$, (*j*) $2\frac{1}{4}$, (*k*) $\frac{50}{72}$.

11. State the prime factors of:

(*a*) 18, (*b*) 26, (*c*) 34, (*d*) 36, (*e*) 40, (*f*) 42,
(*g*) 56, (*h*) 63, (*i*) 70, (*j*) 75, (*k*) 110, (*l*) 132.

B

12. State the L.C.M. of

(*a*) 2, 4, 8; (*b*) 3, 4, 6; (*c*) 12, 18, 24;
(*d*) 20, 30, 40; (*e*) 25, 40, 50; (*f*) 72, 18, 24.

13. What is the length of the side of a square which is equal in area to that of a rectangle which is 25 feet long and 9 feet wide?

14. What is the length of the side of a square which is equal in area to a rectangle 50 feet long and 72 feet wide?

15. How many pieces 7″ long can be cut from a rod which is 12′ 9″ long? What is the length of the piece remaining?

16. Some castings for a lathe were made in a shop. The base weighed 435 lb.; the pulleys 15 lb.; the tailstock 36 lb.; the headstock 62 lb.; the face plate 3 lb.; the bed 215 lb. What was the total weight?

17. A pressure switch on an air pump is set so that the motor begins to operate at 175 lb. and is shut off at 203 lb. By how many pounds does the pressure in the tank vary?

18. It is estimated that it requires 45 minutes to set up the work and cut the teeth on a gear blank. What time should a foreman estimate will be required for cutting 36 gears?

19. (*a*) Divide 97,536 by 193.
(*b*) Multiply the divisor by your quotient and to this product add the remainder.

20. In upholstering each car 438 fasteners are used. It is known that 457,243 have been taken from stock. Of these 256 have been lost or broken. How many cars should have been upholstered?

21. Express each of the following numbers as a product of its prime factors:

(*a*) 90, (*b*) 98, (*c*) 112, (*d*) 128,
(*e*) 156, (*f*) 1,225, (*g*) 729, (*h*) 1,260.

22. Find the square root of each of the following numbers:

(*a*) 225, (*b*) 324, (*c*) 484,
(*d*) 676, (*e*) 784, (*f*) 1,521,
(*g*) 2,916, (*h*) 7,056, (*i*) 12,100.

23. Express each of the following numbers as a power of an integer:

(*a*) 64, (*b*) 1,000, (*c*) 256,
(*d*) 1,728, (*e*) 3,375.

24. Find the Lowest Common Multiple of the following:

(*a*) 75, 45 and 125; (*b*) 27, 81, 108 and 36;
(*c*) 25, 40 and 60; (*d*) 21, 105 and 42.

25. Simplify:

(*a*) $3(5 + 9) - 2(15 - 3)$, (*b*) $\dfrac{546 + 288}{281 - 142}$,

(*c*) $6^2 - (15 - 12)^2$, (*d*) $25 \times 36 - 15 \times 8 + \dfrac{56}{4}$,

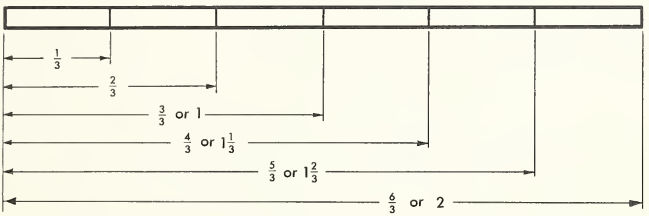
(*e*) $8 - 2 \times 3$, (*f*) $8 \times 9 - 3$.

CHAPTER 4

Fractional Measures

Common Fractions

(1) **A common fraction may be used to express a measure which consists of one or more of the equal parts into which a whole has been divided.**



The integer below the line, which is called the **denominator** of the fraction, indicates the number of equal parts into which the whole was divided.

The integer above the line, which is the **numerator**, indicates the number of unit fractional parts contained in the measure.

A common fraction indicating a measure less than unity is a **proper fraction**; one indicating a measure more than unity is an **improper fraction**.

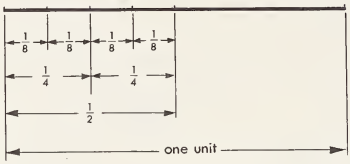
In the diagram above, $\frac{1}{3}$ and $\frac{2}{3}$ are proper fractions; $\frac{4}{3}$ and $\frac{5}{3}$ are improper fractions.

It can be seen that $\frac{4}{3}$ is the same measure as $1\frac{1}{3}$. An improper fraction may be expressed as a mixed number.

(2) **A common fraction may be used to express a quotient.**

For example, if \$3 is to be divided equally among 4 persons, then the amount that each receives in dollars is $3 \div 4 = \frac{3}{4}$.

Fundamental Principle of Fractions



If a unit is divided into 2 equal parts, each of the parts is $\frac{1}{2}$. If this same unit is divided into 4 equal parts, each part is $\frac{1}{4}$. From the diagram it may be seen that

$$\frac{1}{2} = \frac{2}{4}. \text{ Similarly } \frac{1}{2} = \frac{4}{8} \text{ and } \frac{3}{4} = \frac{6}{8}.$$

If you examine each of the above pairs of equivalent fractions, you will notice that the value of a fraction is not altered when the numerator and the denominator of the fraction are each multiplied by the same number—zero excepted.

Looking again at the diagram, notice that

$$\frac{4}{8} = \frac{1}{2} \text{ which is the same as } \frac{4}{8} = \frac{4 \div 4}{8 \div 4},$$

$$\text{and that } \frac{6}{8} = \frac{3}{4} \text{ which is the same as } \frac{6}{8} = \frac{6 \div 2}{8 \div 2}.$$

Evidently the value of a fraction is not changed when the numerator and the denominator are each divided by the same number—zero excepted.

This fundamental principle of fractions may be stated thus:

The value of a fraction is not changed when the numerator and the denominator are each multiplied or divided by the same number—zero excepted.

Example 1:

Express $\frac{3}{4}$ as an equivalent fraction with denominator 32.

Solution: $\frac{3}{4} = \frac{3 \times 8}{4 \times 8}$

$= \frac{24}{32}$.

To obtain the new denominator, 32, the original denominator, 4, must be multiplied by 8.
Therefore, the original numerator, 3, must be multiplied by 8 to obtain the new numerator.

Example 2: Reduce $\frac{38}{95}$ to lowest terms.

Solution: $\frac{38}{95} = \frac{2 \times 19}{5 \times 19}$

$= \frac{2}{5}$.

Express the numerator and the denominator in factored form.
Divide the numerator and the denominator by 19.

The operation of dividing the numerator and the denominator of the fraction by the same number is called **cancellation**.

Exercise

A

1. State the fundamental principle of fractions.

2. State three fractions equivalent to $\frac{1}{3}$.

3. Give four fractions equivalent to $\frac{50}{100}$.

4. Express each of the following fractions as a mixed number:

(*a*) $\frac{12}{5}$, (*b*) $\frac{15}{4}$, (*c*) $\frac{19}{8}$, (*d*) $\frac{72}{16}$, (*e*) $\frac{85}{64}$.

5. Express each of the following measures as an improper fraction:

(*a*) $7\frac{1}{2}$, (*b*) $3\frac{3}{4}$, (*c*) $5\frac{7}{16}$, (*d*) $3\frac{3}{64}$, (*e*) $2\frac{7}{32}$.

6. Change the following fractions to the equivalent number of 16ths:

$\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{8}$, $\frac{3}{8}$, $\frac{7}{8}$, $\frac{3}{4}$.

7. Change to equivalent 32nds: $\frac{3}{8}$, $\frac{3}{16}$, $\frac{5}{16}$, $\frac{7}{8}$, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{13}{16}$, $\frac{11}{16}$.

8. Change to equivalent 64ths: $\frac{5}{16}$, $\frac{13}{16}$, $\frac{7}{8}$, $\frac{3}{4}$, $\frac{11}{32}$, $\frac{17}{32}$, $\frac{1}{2}$, $\frac{3}{32}$.

B

9. Make a drawing to represent a carpenter's six-inch rule in which the first two inches are divided into halves, quarters, eighths and sixteenths.

(*a*) Indicate each of the following measures on the diagram:

(i) $\frac{5}{8}$, (ii) $\frac{7}{16}$, (iii) $\frac{25}{16}$, (iv) $\frac{15}{8}$.

(*b*) Express each of the following measures as quarters plus or minus the smallest possible unit fraction,[1]

(i) $\frac{5}{16}$, (ii) $\frac{7}{8}$, (iii) $\frac{11}{16}$, (iv) $1\frac{13}{16}$.

10. Reduce to fractions with the lowest possible integral numerator and denominator:

(*a*) $\frac{25}{60}$, (*b*) $\frac{288}{648}$, (*c*) $\frac{34}{51}$, (*d*) $\frac{52}{65}$, (*e*) $\frac{7}{91}$,

(*f*) $\frac{64}{96}$, (*g*) $\frac{75}{90}$, (*h*) $\frac{\frac{1}{2}}{\frac{1}{8}}$, (*i*) $\frac{\frac{2}{3}}{\frac{5}{6}}$, (*j*) $\frac{1\frac{1}{2}}{1\frac{1}{4}}$.

11. Change the form of the following fractions so that the denominators in each case will be 12:

(*a*) $\frac{3}{4}$, (*b*) $\frac{5}{6}$, (*c*) $\frac{12}{24}$, (*d*) $\frac{15}{18}$, (*e*) $\frac{150}{100}$.

C

12. Reduce the following fractions to lowest terms:

(*a*) $\frac{75}{85}$, (*b*) $\frac{51}{85}$, (*c*) $\frac{575}{625}$, (*d*) $\frac{441}{784}$, (*e*) $\frac{324}{1296}$,

(*f*) $\frac{121}{1331}$, (*g*) $\frac{72}{1728}$, (*h*) $\frac{4025}{5625}$, (*i*) $\frac{198}{9801}$.

[1] A unit fraction has a numerator of 1.

13. Complete the following statements:

(*a*) $\frac{3}{7} = \frac{\quad}{98}$. (*b*) $\frac{7}{9} = \frac{\quad}{127}$. (*c*) $\frac{7}{13} = \frac{\quad}{91}$.

(*d*) $\frac{4}{11} = \frac{84}{\quad}$. (*e*) $\frac{21}{25} = \frac{273}{\quad}$. (*f*) $\frac{19}{24} = \frac{361}{\quad}$.

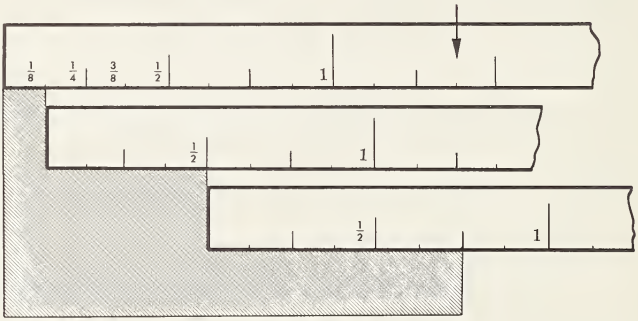
(*g*) $\frac{24}{36} = \frac{\quad}{48}$. (*h*) $\frac{18}{24} = \frac{\quad}{100}$. (*i*) $\frac{2}{3} = \frac{\quad}{10}$.

14. Express the following fractions with a denominator of 100:

(*a*) $\frac{3}{4}$, (*b*) $\frac{1}{8}$, (*c*) $\frac{3}{8}$, (*d*) $\frac{5}{8}$, (*e*) $\frac{1}{3}$, (*f*) $\frac{1}{7}$,
(*g*) $\frac{1}{200}$, (*h*) $\frac{1}{300}$, (*i*) $\frac{1}{800}$, (*j*) $\frac{3}{200}$, (*k*) $\frac{7}{300}$.

Addition and Subtraction of Common Fractions

In the drawing below, what is the over-all length of the shaded portion?



Note: Scale 2″ represents 1″. This is usually indicated thus: Scale 2:1

The over-all length in inches is $\frac{1}{8} + \frac{1}{2} + \frac{3}{4}$

$$= \frac{1 + 4 + 6}{8}$$

$$= \frac{11}{8}$$

$$= 1\frac{3}{8}.$$

To simplify a measure containing addition and subtraction of fractions

(*i*) find a suitable number, the L.C.M., that can be used as a common denominator;

(*ii*) express each fraction as an equivalent fraction having this common denominator;

(*iii*) accurately perform the required addition and subtraction of the new numerators.

Example 1: Simplify $\frac{5}{24} + \frac{7}{30} + \frac{2}{45}$

Solution:

$$\frac{5}{24} + \frac{7}{30} + \frac{2}{45}$$

$$= \frac{5 \times 15}{24 \times 15} + \frac{7 \times 12}{30 \times 12} + \frac{2 \times 8}{45 \times 8}$$

$$= \frac{75 + 84 + 16}{360}$$

$$= \frac{175}{360}$$

$$= \frac{35}{72}.$$

$24 = 2 \times 2 \times 2 \times 3$
$30 = 2 \times 3 \times 5$
$45 = 3 \times 3 \times 5$
L.C.M. of 24, 30 and 45 is
$2 \times 2 \times 2 \times 3 \times 5 \times 3$
$= 360$

$360 = 24 \times 15$
$360 = 30 \times 12$
$360 = 45 \times 8.$

Example 2: Simplify $\frac{7}{15} - \frac{13}{25} + \frac{3}{20}$.

Solution:

$$\frac{7}{15} - \frac{13}{25} + \frac{3}{20}$$

$$= \frac{140 - 156 + 45}{300}$$

$$= \frac{185 - 156}{300}$$

$$= \frac{29}{300}.$$

$15 = 3 \times 5$
$25 = 5 \times 5$
$20 = 2 \times 2 \times 5$

L.C.M. of 15, 25 and 20 is
$3 \times 5 \times 5 \times 2 \times 2 = 300$

$300 = 15 \times 20$
$300 = 25 \times 12$
$300 = 20 \times 15$

Exercise

A

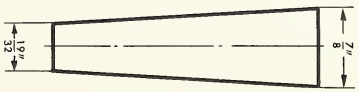
1. Simplify:

(*a*) $\frac{1}{2} + \frac{1}{4}$, (*b*) $\frac{3}{4} + \frac{1}{2}$, (*c*) $\frac{1}{8} + \frac{1}{2}$,
(*d*) $\frac{1}{3} + \frac{1}{2}$, (*e*) $\frac{1}{6} + \frac{1}{2}$, (*f*) $\frac{2}{3} + \frac{2}{3}$,
(*g*) $\frac{1}{8} + \frac{1}{4}$, (*h*) $\frac{3}{4} - \frac{1}{8}$, (*i*) $\frac{2}{3} - \frac{1}{2}$,
(*j*) $\frac{4}{5} - \frac{2}{5}$, (*k*) $\frac{4}{5} - \frac{1}{10}$, (*l*) $\frac{1}{7} - \frac{1}{8}$.

2. In the following pairs of fractions state which is the larger:

(*a*) $\frac{3}{7}$ or $\frac{11}{28}$, (*b*) $\frac{5}{6}$ or $\frac{6}{7}$, (*c*) $\frac{5}{12}$ or $\frac{17}{36}$,
(*d*) $\frac{49}{64}$ or $\frac{3}{4}$, (*e*) $\frac{5}{8}$ or $\frac{39}{64}$, (*f*) $\frac{11}{16}$ or $\frac{45}{64}$,
(*g*) $\frac{29}{32}$ or $\frac{57}{64}$, (*h*) $\frac{1}{7}$ or $\frac{9}{64}$, (*i*) $\frac{5}{7}$ or $\frac{11}{16}$.

3. A tapered pin has end diameters as illustrated. Find the difference between the diameters.



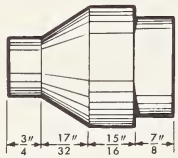
4. Find the inside diameter when the thickness is

(*a*) $\frac{1}{16}''$, (*b*) $\frac{1}{8}''$,
(*c*) $\frac{3}{32}''$, (*d*) $\frac{5}{64}''$.

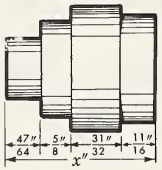
B

5. Find the perimeter of a triangle the sides of which measure $\frac{3}{4}$ in., $\frac{5}{16}$ in. and $\frac{55}{64}$ in. respectively.

6. Find the over-all length of this gland.



7. In the drawing below, find the over-all length indicated by x''.



8. Calculate the lowest common multiple for each of the following sets of numbers:

(*a*) 25, 35 and 45; (*b*) 28, 42 and 54;
(*c*) 34, 51 and 85; (*d*) 18, 24 and 27.

9. Add each of the following and reduce to its lowest terms:

(*a*) $\frac{3}{8} + \frac{5}{16} + \frac{1}{8}$, (*b*) $\frac{5}{8} + \frac{7}{32} + \frac{1}{4} + \frac{7}{16}$,
(*c*) $\frac{21}{32} + \frac{5}{64} + \frac{13}{16} + \frac{1}{2}$, (*d*) $\frac{7}{8} + \frac{7}{16} + \frac{17}{32} + \frac{31}{64}$,
(*e*) $\frac{15}{16} + \frac{5}{8} + \frac{3}{32} + \frac{7}{16}$, (*f*) $\frac{9}{16} + \frac{35}{64} + \frac{17}{32} + \frac{7}{8}$.

10. Simplify:

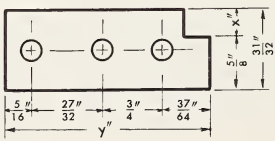
(*a*) $\frac{3}{4} + \frac{5}{6} + \frac{2}{3}$, (*b*) $\frac{5}{8} + \frac{13}{16} - \frac{7}{8}$,
(*c*) $\frac{1}{4} - \frac{1}{2} + \frac{7}{8}$, (*d*) $\frac{7}{15} + \frac{7}{12}$.

11. Simplify:

(*a*) $\frac{5}{16} + \frac{5}{24} - \frac{7}{40}$, (*b*) $\frac{5}{6} - (\frac{1}{3} + \frac{1}{6})$,
(*c*) $\frac{3}{4} - (\frac{2}{3} - \frac{1}{6})$, (*d*) $\frac{5}{18} - \frac{11}{24} + \frac{25}{27}$,
(*e*) $\frac{9}{25} + \frac{16}{35} + \frac{4}{45}$, (*f*) $\frac{19}{28} - \frac{5}{42} - \frac{1}{54}$.

12. For the template illustrated, find

(*a*) the over-all length y,
(*b*) the measure represented by x.



13. Simplify:

(*a*) $\frac{17}{32} - \frac{1}{4} + \frac{5}{8} - \frac{1}{2}$, (*b*) $\frac{63}{64} - \frac{7}{8} + \frac{31}{32} - \frac{15}{16}$,

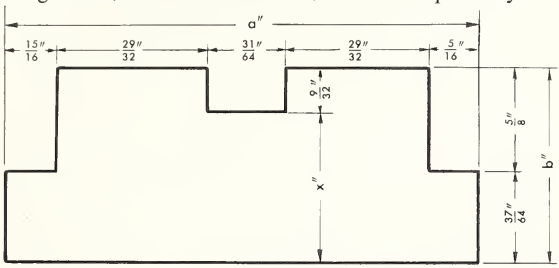
(*c*) $\frac{5}{8} + \frac{31}{64} - \frac{3}{16} - \frac{7}{32} - \frac{5}{64}$, (*d*) $\frac{3}{4} - \frac{7}{20} + \frac{8}{15} + \frac{11}{30}$.

14. Simplify:

(*a*) $\frac{7}{8} + \frac{13}{18} - \frac{7}{24} + \frac{11}{16}$, (*b*) $\frac{9}{12} + \frac{31}{30} - \frac{3}{5} - \frac{1}{4} + \frac{13}{24}$,

(*c*) $\frac{19}{22} - \frac{1}{33} + \frac{5}{6} - \frac{3}{4} - \frac{17}{22}$, (*d*) $\frac{21}{40} + \frac{3}{5} - \frac{5}{16} + \frac{7}{24} - \frac{11}{15}$.

15. In the drawing below, find the measures *a*, *b* and *x* respectively.



16. A piece of flat steel has uniform thickness of $\frac{13}{16}$". A cut $\frac{3}{32}$" is taken off. What additional depth of cut will be required to reduce the thickness to $\frac{43}{64}$"?

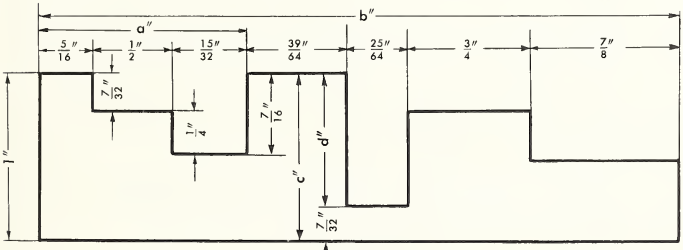
C

17. Arrange the following fractions in order of size beginning with the largest:

(*a*) $\frac{1}{2}, \frac{9}{16}, \frac{17}{32}, \frac{31}{64}$, (*b*) $\frac{4}{5}, \frac{5}{6}, \frac{9}{24}, \frac{11}{12}$,

(*c*) $\frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{7}{10}, \frac{5}{12}, \frac{7}{15}$, (*d*) $\frac{3}{7}, \frac{2}{5}, \frac{4}{9}$.

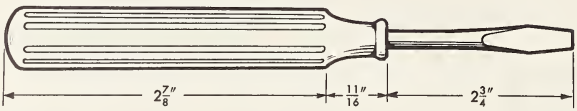
18. In the accompanying diagram, find the measures represented by *a*, *b*, *c* and *d*.



Addition and Subtraction of Mixed Numbers

When you understand thoroughly the principles involved in addition and subtraction of measures less than 1, you can easily extend them to similar operations with mixed numbers. The following examples illustrate suitable procedures.

Example 1: Find the over-all length of this screwdriver.



Solution:

The over-all length in inches is $2\frac{7}{8} + \frac{11}{16} + 2\frac{3}{4}$

$= 2\frac{14}{16} + \frac{11}{16} + 2\frac{12}{16}$

$= 4\frac{37}{16}$

$= 6\frac{5}{16}$.

Example 2: Simplify $1\frac{3}{4} + 9\frac{5}{7} + 3\frac{1}{4} + 4\frac{3}{14}$.

In this case it is advisable to group certain terms, simplify each group and then combine.

Solution: $1\frac{3}{4} + 9\frac{5}{7} + 3\frac{1}{4} + 4\frac{3}{14}$

$= 1\frac{3}{4} + 3\frac{1}{4} + 9\frac{5}{7} + 4\frac{3}{14}$

$= 4\frac{4}{4} + 9\frac{10}{14} + 4\frac{3}{14}$

$= 5 + 13\frac{13}{14}$

$= 18\frac{13}{14}$.

Example 3: Simplify $9\frac{7}{8} - 2\frac{1}{2}$.

Solution: $9\frac{7}{8} - 2\frac{1}{2}$

$= 9\frac{7}{8} - 2\frac{4}{8}$

$= 7\frac{3}{8}$.

Or $9\frac{7}{8} - 2\frac{1}{2}$

$= (9 + \frac{7}{8}) - (2 + \frac{1}{2})$

$= 9 - 2 + \frac{7}{8} - \frac{1}{2}$

$= 7 + \frac{7}{8} - \frac{4}{8}$

$= 7\frac{3}{8}$.

Example 4: Simplify $9\frac{1}{2} - 2\frac{7}{8}$.

Solution: $9\frac{1}{2} - 2\frac{7}{8}$

$= 9\frac{4}{8} - 2\frac{7}{8}$

$= 8\frac{12}{8} - 2\frac{7}{8}$

$= 6\frac{5}{8}$.

In this example the fraction in the subtrahend is larger than the fraction in the minuend. Adjust the minuend by changing 1 unit to fractional form to increase the minuend's fractional part.

With practice you can do most subtractions of this type, mentally, with the following method:

$9\frac{1}{2} - 2\frac{7}{8}$

$= 9\frac{1}{2} - 3 + \frac{1}{8}$

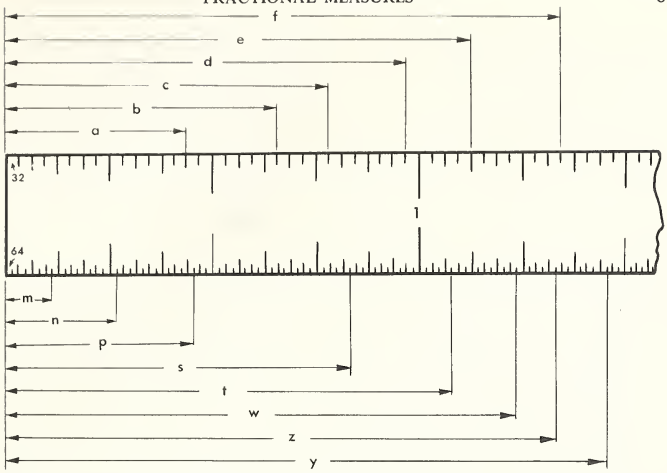
$= 6\frac{5}{8}$.

Since $2\frac{7}{8}$ is close to the integer 3, subtract 3 and then adjust by adding $\frac{1}{8}$.

Exercise

A

1. The following drawing is a large scale illustration to represent a steel rule graduated in 32nds and 64ths of an inch respectively. Study the drawing and state the measure represented by each of the letters *a*, *b*, *c*, *d*, etc. on the illustration.



2. What is the resulting measure when $\frac{1}{32}$ is added to each of the following?

(*a*) $\frac{1}{2}$, (*b*) $\frac{3}{4}$, (*c*) $1\frac{5}{16}$, (*d*) $5\frac{1}{8}$, (*e*) $3\frac{7}{8}$.

3. What is the resulting measure when $\frac{1}{32}$ is subtracted from each of the following?

(*a*) $\frac{1}{4}$, (*b*) $2\frac{5}{8}$, (*c*) $5\frac{7}{16}$, (*d*) $6\frac{9}{16}$, (*e*) $1\frac{1}{64}$.

4. What is the resulting measure when $\frac{1}{64}$ is added to each of the following?

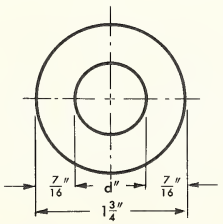
(*a*) $\frac{1}{8}$, (*b*) $\frac{1}{16}$, (*c*) $7\frac{3}{32}$, (*d*) $5\frac{1}{2}$, (*e*) $3\frac{17}{32}$.

5. What is the resulting measure when $\frac{1}{64}$ is subtracted from each of the following?

(*a*) $\frac{1}{2}$, (*b*) 2, (*c*) $3\frac{1}{4}$, (*d*) $5\frac{1}{16}$, (*e*) $8\frac{3}{4}$.

B

6. From the drawing, determine the diameter of the inner circle.



Simplify each of the following expressions:

7. $3\frac{5}{8} - 2\frac{15}{16} + 5\frac{3}{4}$,

8. $2\frac{1}{2} - 5\frac{1}{4} + 9\frac{5}{8}$,

9. $(3\frac{1}{2} + 5\frac{1}{8}) - (4\frac{1}{8} - 3\frac{5}{16})$,

10. $5\frac{7}{15} + 3\frac{4}{9} + 4\frac{5}{18}$,

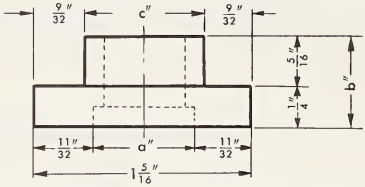
11. $3\frac{7}{12} - 5\frac{7}{8} + 9\frac{2}{3} + \frac{5}{6}$,

12. $5 - (3\frac{1}{4} - 1\frac{7}{8})$,

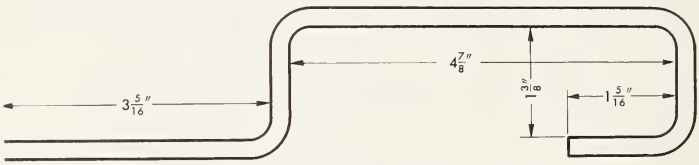
13. $253\frac{3}{8} + 45\frac{7}{16}$,

14. $386\frac{7}{24} - 156\frac{13}{15}$.

15. Find the dimensions a, b and c indicated in the drawing below.



16. Find the length of $\frac{1}{8}''$ thick strap iron required to bend into the handle shown. The allowance for bending for each turn is $\frac{1}{2}$ of the thickness of the metal.



17. The carbon for the lamp of a blueprint machine is $9\frac{1}{2}''$ long. Every 24 minutes $\frac{3}{8}''$ burned off. What length of carbon will remain after the machine has operated continuously for 1 hour?

18. A rough casting weighs $8\frac{5}{16}$ lb. When it is finished it weighs $7\frac{5}{8}$ lb. How much material was machined off?

19. A margin of $\frac{5}{8}''$ is drawn on all four sides of a piece of note-paper $8\frac{1}{2}''$ by $11''$. What is the length and width of the new rectangle?

20. If we allow $\frac{1}{16}$ inch for each cut, what is the minimum length of strap iron from which we can cut three pieces of lengths $1\frac{3}{8}''$, $5\frac{5}{16}''$ and $7\frac{1}{4}''$?

21. The total resistance of three lights in series is obtained by adding their separate resistances. Find the total resistance of a circuit with three lights which have resistances of $3\frac{1}{2}$ ohms, $4\frac{2}{3}$ ohms and $7\frac{3}{10}$ ohms respectively.

22. What is the difference in temperature between $68\frac{3}{10}$ degrees and $57\frac{7}{10}$ degrees?

C

23. If $\frac{1}{8}''$ is allowed for each cut, what will be the minimum length of a steel bar to provide the following pieces: $2\frac{3}{16}''$, $3\frac{7}{8}''$, $1\frac{3}{32}''$ and $4\frac{1}{4}''$?

24. From a copper bar $21\frac{1}{2}''$ long the following four pieces are cut: $2\frac{1}{8}''$, $3\frac{3}{4}''$, $1\frac{15}{32}''$ and $3\frac{9}{16}''$. If $\frac{1}{16}''$ is allowed for each cut, what will be the remaining length of the bar?

Multiplication of an Integer by a Fraction

Example: Find the value of $5 \times \frac{3}{4}$.

Solution: $5 \times \frac{3}{4}$ means $\frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4}$.

$$\therefore 5 \times \frac{3}{4} = \frac{15}{4}$$
$$= 3\frac{3}{4}.$$

To multiply a fraction by an integer, multiply the numerator by the integer and keep the same denominator.

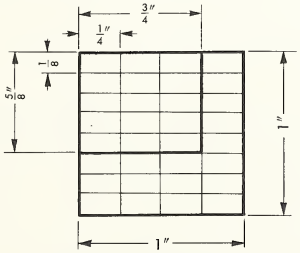
Multiplication of Two Fractions

A simple problem will illustrate the method used to find the product of two fractions.

Find the area of a rectangle $\frac{3}{4}$ inch long by $\frac{5}{8}$ inch wide.

Area of the rectangle in sq. in. is $\frac{3}{4} \times \frac{5}{8}$.

This product could be found by making a diagram to illustrate the situation. Since the unit of measurement is 1 square inch, extend the length and width of the rectangle to 1 inch, thus forming a square inch. Mark off the length in $\frac{1}{4}$-inch parts because there are 3 quarter-inch parts in it. Similarly mark off the width into $\frac{1}{8}$-inch parts:



As there are 32 small rectangles in this square inch, the measure of the area of each of these in sq. in. is $\frac{1}{32}$. It will be observed that this denominator, 32, is the product of the denominators of the fractions to be multiplied.

The number of these small rectangles in our area is $3 \times 5 = 15$. Therefore, the area of the rectangle in sq. in. is $\frac{15}{32}$.

Hence $\frac{3}{4} \times \frac{5}{8} = \frac{15}{32}$.

The product of two fractions is a fraction having for its numerator the product of the numerators of the original fractions and having for its denominator the product of the denominators of the original fractions.

Example 1: Find the product of $\frac{8}{15}$ and $\frac{9}{10}$.

Solution:

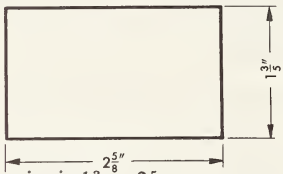
$$\frac{8}{15} \times \frac{9}{10} = \frac{\overset{4}{\cancel{8}} \times \overset{3}{\cancel{9}}}{\underset{5}{\cancel{15}} \times \underset{5}{\cancel{10}}}$$

Express the respective products. Reduce the resulting fraction to its lowest terms.

$$= \frac{12}{25}.$$

Example 2: Calculate the area of a rectangle $1\frac{3}{5}$ inches wide by $2\frac{5}{8}$ inches long.

Solution:



Area of rectangle in sq. in. is $1\frac{3}{5} \times 2\frac{5}{8}$

$$= \frac{\overset{1}{\cancel{8}}}{5} \times \frac{21}{\underset{1}{\cancel{8}}}$$

Change the mixed numbers to improper fractions.

$$= \frac{21}{5}$$

$$= 4\tfrac{1}{5}.$$

Operations Involving "of"

The operation involving "of" can be performed by using multiplication.

To find $\frac{2}{3}$ of 12, we might proceed as follows:

$\frac{1}{3}$ of 12 is 4
$\frac{2}{3}$ of 12 is $2 \times 4 = 8$.

This could have been done more concisely in the following way:

$$\tfrac{2}{3} \text{ of } 12 = \tfrac{2}{3} \times 12$$
$$= 8.$$

Exercise

A

1. Simplify:

(*a*) $\frac{1}{2} \times \frac{1}{3}$, (*b*) $\frac{2}{3} \times \frac{3}{4}$, (*c*) $\frac{1}{2}$ of $\frac{2}{3}$,
(*d*) $\frac{2}{3}$ of $\frac{7}{10}$, (*e*) $\frac{5}{8} \times \frac{4}{5}$, (*f*) $\frac{7}{8}$ of $1\frac{1}{2}$,
(*g*) $1\frac{1}{2} \times 1\frac{1}{2}$, (*h*) $5\frac{1}{2} \times 5\frac{1}{2}$, (*i*) $\frac{3}{10}$ of $\frac{5}{8}$.

2. What is the volume of a cube each side of which is

(*a*) $\frac{1}{2}''$, (*b*) $\frac{1}{3}''$, (*c*) $\frac{3}{4}''$, (*d*) $1\frac{1}{2}''$?

3. Simplify:

(*a*) $\frac{3}{4} \times 20$, (*b*) $\frac{1}{2} \times 13$, (*c*) $\frac{2}{3} \times 18$,
(*d*) $12 \times 1\frac{1}{2}$, (*e*) $24 \times \frac{3}{8}$, (*f*) $28 \times \frac{4}{7}$,
(*g*) $\frac{2}{3} \times 25$, (*h*) $\frac{7}{8} \times 36$, (*i*) $\frac{3}{5}$ of 65,
(*j*) $\frac{1}{8}$ of 100, (*k*) $\frac{1}{2}$ of $5\frac{1}{2}$, (*l*) $\frac{2}{3}$ of 129.

4. A certain drawing is to be made so that $\frac{1}{2}''$ will represent 1 foot. What lengths of lines will be used for the following measurements:

(*a*) 6′, (*b*) 12′ 6″, (*c*) 9″, (*d*) 5′ 3″?

5. In a certain drawing $\frac{3}{8}''$ represents 1 foot. What length of line will be used to represent $6\frac{1}{2}$ ft.?

B

6. Simplify the following expressions:

(*a*) $5\frac{1}{2} \times 3\frac{5}{11}$, (*b*) $\frac{2}{3} \times \frac{5}{8} \times \frac{9}{10}$,
(*c*) $3\frac{1}{7} \times 3\frac{3}{4}$, (*d*) $3\frac{1}{7} \times 3\frac{1}{2} \times 3\frac{1}{2}$,
(*e*) $2(3\frac{1}{2} + 4\frac{1}{4})$, (*f*) $(5\frac{1}{2})^2$,
(*g*) $2\frac{1}{2} \times \frac{3}{5} + \frac{3}{8} \times \frac{4}{9}$, (*h*) $\frac{3}{4}$ of $8 - \frac{3}{5}$,
(*i*) $\frac{5}{9}(36\frac{1}{2} - 32)$, (*j*) $(4\frac{1}{2} + 3\frac{1}{2})(5\frac{1}{2} - 2\frac{3}{4})$.

7. One cubic foot is equivalent to approximately $6\frac{1}{4}$ gallons. How many gallons will be held by each of the tanks having the following volumes?

(*a*) $8\frac{4}{5}$ cubic feet, (*b*) $5\frac{7}{9}$ cubic feet, (*c*) $16\frac{2}{3}$ cubic feet.

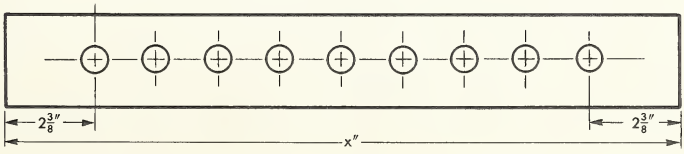
8. A rod is $16\frac{1}{2}$ ft. in length. How many feet are in $3\frac{1}{3}$ rods; in $\frac{1}{4}$ rod; in $1\frac{1}{2}$ rods?

9. The circumference of a circle is approximately $3\frac{1}{7}$ times the diameter. Find the circumference of each of the circles having the following diameters:

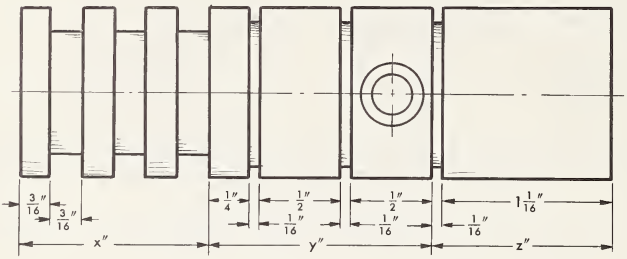
(*a*) $2\frac{1}{2}$ feet; (*b*) $3\frac{1}{2}$ feet; (*c*) $5\frac{1}{4}$ inches.

10. Calculate the volume of a rectangular solid the length of which is $6\frac{2}{3}$ in., the width $2\frac{1}{4}$ in., and the height $3\frac{1}{2}$ in.

11. In the drawing below, (*a*) express the measure x of the over-all length of the bar in terms of the measures given; (*b*) simplify the expression obtained in part (*a*). Assume that the holes are equally spaced $1\frac{3}{16}''$ apart.



12. (*a*) On the drawing of a piston calculate the measures represented by x, y and z respectively.



(*b*) Indicate the measure of the over-all length of the piston by an arithmetical expression containing the given measures. Simplify this expression.

13. (*a*) A rectangle is $3\frac{5}{16}$ inches long and $2\frac{3}{8}$ inches wide. What is its perimeter?

(*b*) If four of these rectangles are placed together to form one large rectangle, what will be the perimeter of the rectangle thus formed?

14. What is the volume of a cube each side of which is $\frac{7}{8}$ inches long?

C

15. A 20′ steel bar $\frac{1}{2}''$ in diameter had the following pieces cut off: 2 pieces $17\frac{3}{4}''$ long, 3 pieces $6\frac{1}{8}''$ long and 7 pieces $23\frac{1}{2}''$ long. If we allow $\frac{1}{16}''$ for each cut, what was the length of the bar left?

16. A rush order is received for 250 pins. Each pin must be $4\frac{3}{32}''$ long. The width of the cutting tool is $\frac{1}{8}''$. How many lengths of 10′ stock must be purchased to fill the order?

17. An apprentice is asked to make 68 door stoppers. The actual time taken to make one stopper is $18\frac{3}{4}$ minutes. An additional $\frac{5}{16}$ of a minute is required to put the job in and take it out of the machine. What will be the labour cost if the apprentice receives $1·50 per hour?

Division of Fractions

Division can be used for either of two purposes:

(1) to obtain the number of times one quantity is contained in another;

(2) to obtain the measure of one of the parts of a quantity when it is divided into a number of equal parts.

Example 1: How many pieces of wire $\frac{3}{8}''$ long can be obtained from a wire 6″ long?

Solution 1: The number of pieces is $6 \div \frac{3}{8}$.
The measure 6 could be expressed as 48 eighths. Then, the length of the wire, in eighths of inches, is 48. The length of 1 piece of wire, in eighths of inches, is 3. Therefore, the number of pieces is $\frac{48}{3} = 16$.
Evidently to divide 6 by $\frac{3}{8}$ we multiply 6 by 8 and divide the product by 3.

$$\text{i.e., } 6 \div \tfrac{3}{8} = 6 \times \tfrac{8}{3}$$
$$= 16.$$

This case illustrates that **the operation of division can be accomplished by inverting the divisor and then multiplying.**

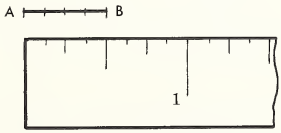
Solution 2: The number of pieces is $6 \div \frac{3}{8}$

$$= \frac{6}{\frac{3}{8}}$$
$$= \frac{6 \times 8}{\frac{3}{8} \times 8} \qquad \text{Using the fundamental principle of fractions}$$
$$= 6 \times \tfrac{8}{3}$$
$$= 16.$$

Evidently $6 \div \frac{3}{8}$ is equivalent to $6 \times \frac{8}{3}$.

The operation of division can be accomplished by inverting the divisor and then multiplying.

Example 2: A wire $\frac{1}{2}''$ long is marked off into 4 equal parts. How long is each of the parts?
Solution:



The length of each part in inches is $\frac{1}{2} \div 4$

$$= \frac{\frac{1}{2} \times 2}{4 \times 2}$$
$$= \tfrac{1}{8}.$$

It can be seen from the diagram that each part of AB is $\frac{1}{8}''$ long.

To divide, invert the divisor and multiply.

Example 3: Find the length of a rectangle if its area is $\frac{5}{8}$ sq. in. and its width is $\frac{3}{4}$ in.
Solution: The length of the rectangle in inches is $\frac{5}{8} \div \frac{3}{4}$

$$= \tfrac{5}{8} \times \tfrac{4}{3}$$
$$= \tfrac{5}{6}.$$

Example 4: A man drives 65 miles in $1\frac{3}{4}$ hours. What is his average speed?

Solution: The average speed in miles per hour is $65 \div 1\frac{3}{4}$

$$= 65 \div \tfrac{7}{4}$$
$$= 65 \times \tfrac{4}{7}$$
$$= \frac{260}{7}$$
$$= 37\tfrac{1}{7}.$$

Exercise

A

1. Simplify the following expressions:

(*a*) $\frac{1}{2} \div 2$, (*b*) $3 \div \frac{1}{2}$,
(*c*) $16 \div \frac{1}{4}$, (*d*) $\frac{1}{8} \div 2$,
(*e*) $\frac{2}{3} \div 3$, (*f*) $\frac{1}{2} \div 8$,
(*g*) $\frac{5}{8} \div \frac{3}{4}$, (*h*) $\frac{1}{4} \div 16$,
(*i*) $\frac{1}{2} \div \frac{1}{2}$, (*j*) $3 \div \frac{1}{3}$,
(*k*) $\frac{1}{2} \div \frac{1}{4}$, (*l*) $1\frac{1}{2} \div 1\frac{1}{4}$,
(*m*) $8\frac{3}{4} \div 2$, (*n*) $5\frac{1}{2} \div \frac{1}{2}$.

2. Simplify:

(*a*) $\dfrac{\frac{3}{4}}{\frac{1}{2}}$, (*b*) $\dfrac{\frac{5}{16}}{\frac{3}{8}}$, (*c*) $\dfrac{1}{\frac{1}{8}}$,

(*d*) $\dfrac{8}{\frac{1}{2}}$, (*e*) $\dfrac{6\frac{1}{2}}{1\frac{1}{2}}$, (*f*) $\dfrac{3\frac{1}{4}}{2\frac{1}{2}}$.

B

3. Simplify:

(*a*) $4 \div \frac{3}{4}$, (*b*) $\frac{15}{22} \div 5$, (*c*) $3\frac{1}{2} \div 7$,
(*d*) $\frac{3}{4} \div \frac{2}{3}$, (*e*) $1\frac{3}{4} \div 10\frac{1}{2}$, (*f*) $2\frac{1}{2} \div 6\frac{1}{4}$,
(*g*) $6\frac{1}{4} \div 2\frac{1}{2}$, (*h*) $3\frac{1}{3} \div \frac{1}{3}$, (*i*) $\frac{1}{3} \div 3\frac{1}{3}$.

4. Simplify:

(*a*) $7 \div \frac{1}{5}$, (*b*) $6 \div \frac{2}{3}$, (*c*) $\frac{1}{2} \div 5$,
(*d*) $1\frac{1}{2} \div 100$, (*e*) $\frac{3}{4} \div 100$, (*f*) $100 \div \frac{3}{8}$.

5. A rod 6 inches long is divided into $\frac{3}{4}$-inch pieces. How many such pieces are there?

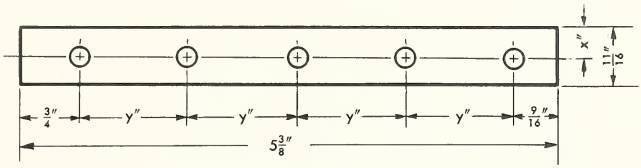
6. A wire $\frac{15}{16}$ inches long is marked off into 30 equal divisions. How long is each division?

7. If a drill advances into the work $\frac{1}{64}''$ each turn, how many turns are necessary to advance through 3″ of material?

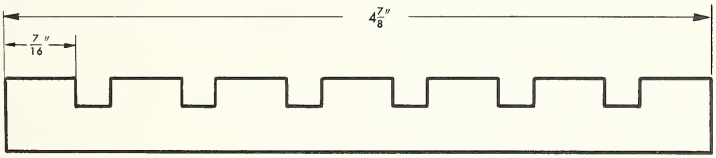
8. If $1\frac{1}{2}$ tons of coal cost \$33·00, what should be the cost of $6\frac{1}{2}$ tons of this coal?

9. For making nails, pieces of wire $2\frac{3}{4}''$ long are required. How many such pieces can be cut from a roll of wire 100 ft. long? What would be the calculated length of the piece remaining after the full length pieces are cut off?

10. (*a*) Find the measure of the distance y between centres of equally spaced holes in the diagram.
(*b*) What measure does x represent?



11. If each of the flutes in the accompanying diagram is $\frac{7}{16}''$, find the width of the equal open spaces.



Complex Fractions

A **complex fraction** is one in which the terms are not integers.

Example: Simplify: $\dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{3}{4}-\frac{1}{8}}$.

Solution:

Simplify the numerator and denominator of the complex fraction before the division is performed.

$$\frac{\frac{1}{2}+\frac{1}{3}}{\frac{3}{4}-\frac{1}{8}} = \frac{\frac{5}{6}}{\frac{5}{8}}$$

$$= \frac{\cancel{5}}{\cancel{6}} \times \frac{\cancel{8}^{4}}{\cancel{5}}$$

$$= \frac{4}{3} = 1\frac{1}{3}.$$

Exercise

A

1. Simplify:

(*a*) $\dfrac{1}{\frac{3}{4}}$, (*b*) $\dfrac{\frac{1}{2}}{\frac{1}{4}}$, (*c*) $\dfrac{1\frac{1}{2}}{\frac{1}{2}}$,

(*d*) $\dfrac{\frac{1}{2}+\frac{1}{4}}{\frac{1}{2}-\frac{1}{4}}$, (*e*) $\dfrac{\frac{1}{2}\times\frac{1}{4}}{\frac{1}{16}}$, (*f*) $\dfrac{1\frac{3}{4}}{2}$,

(*g*) $\dfrac{3-\frac{1}{6}}{\frac{1}{2}}$, (*h*) $\dfrac{3+\frac{1}{2}\times 4}{2-1\frac{1}{2}}$, (*i*) $\dfrac{5-\frac{1}{3}}{2}$.

B

2. A steel bar is known to weigh $2\frac{1}{8}$ lb. per foot. What is the length of a piece weighing $8\frac{1}{4}$ lb.?

3. A box contains bolts each of which weighs $\frac{1}{4}$ lb. If the box of bolts weighs $60\frac{1}{4}$ lb. and the box when empty weighs $4\frac{1}{4}$ lb., find the number of bolts in the box.

4. Simplify the following expressions:

(*a*) $\dfrac{\frac{1}{2}+\frac{3}{4}}{\frac{5}{8}-\frac{3}{8}}$, (*b*) $\dfrac{3\frac{1}{2}+5\frac{1}{8}}{3\frac{1}{2}\times 5\frac{1}{8}}$.

5. Simplify:

(*a*) $\dfrac{4\times 3\frac{1}{7}\times(\frac{2}{3})^2}{3}$, (*b*) $\dfrac{\frac{3}{4}+\frac{5}{8}}{3\times\frac{4}{5}}$, (*c*) $\dfrac{\frac{6}{7}\times\frac{5}{6}+2\frac{1}{3}}{\frac{1}{8}\times\frac{2}{3}\times\frac{6}{7}}$, (*d*) $\dfrac{(\frac{3}{5}\times\frac{2}{3})^2+(\frac{3}{5})^2}{2\times\frac{3}{5}}$.

6. Simplify:

(*a*) $\frac{3}{5}$ of $6\frac{2}{3}-\frac{2}{5}\div\frac{1}{2}\times 1\frac{2}{3}$, (*b*) $\frac{1}{2}(3\frac{1}{2}+5\frac{3}{4})5\frac{1}{2}$,

(*c*) $\frac{2}{3}+2\times\frac{5}{8}\times\frac{4}{9}+\frac{4}{15}$, (*d*) $7\frac{3}{32}-(2\times\frac{5}{16}+\frac{3}{8})$.

C

7. In calculating the resistance of a parallel circuit, the following expressions occurred. Simplify them.

(*a*) $\dfrac{1}{\frac{1}{8}}+\dfrac{1}{\frac{1}{4}}$, (*b*) $\dfrac{1}{\frac{3}{4}}+\dfrac{1}{\frac{6}{10}}$, (*c*) $\dfrac{1}{3\frac{1}{4}}+\dfrac{1}{2\frac{1}{2}}$.

8. Each of two rectangles has an area of $7\frac{1}{5}$ sq. in. One has a width of $1\frac{1}{2}$ inches; the other, $2\frac{1}{4}$ inches.

(*a*) Express in terms of the given data the measure of the difference in their lengths. Simplify this expression.

(*b*) Express the difference in their perimeters in terms of the given data. Simplify this expression.

9. In calculating the resistance of a certain series-parallel electrical circuit, the following expression was obtained:

$$\frac{1}{\frac{1}{\frac{1}{2}}+\frac{1}{\frac{1}{16}}}+\frac{1}{\frac{1}{\frac{1}{4}}+\frac{1}{\frac{1}{8}}}.$$

Simplify this expression.

10. Simplify these fractional expressions:

(*a*) $\dfrac{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{\frac{2}{5}+\frac{2}{3}+\frac{2}{9}}$,

(*b*) $\dfrac{\frac{3}{4}-\frac{1}{2}}{\frac{1}{2}+\frac{1}{3}}$,

(*c*) $\frac{5}{8}$ of $\frac{4}{5} \div \frac{7}{20}$,

(*d*) $\frac{20}{27} \div \frac{1}{2}$ of $\frac{4}{9}$,

(*e*) $\dfrac{2\frac{1}{4} \div \frac{3}{5}}{7\frac{1}{2}-3\frac{5}{6}}$,

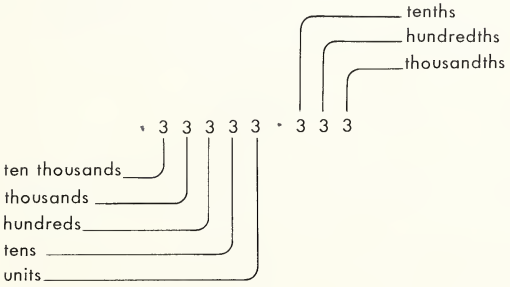
(*f*) $1+\dfrac{1}{\frac{2}{3}+\frac{2}{5}}$,

(*g*) $2+\dfrac{1}{2+\dfrac{1}{3+\frac{1}{4}}}$,

(*h*) $\dfrac{1}{3+\dfrac{1}{5\frac{1}{7}}}$.

Decimal Numeration and Notation

The simplicity of our number system is due to the fact that the "value" represented by a digit depends upon its position in the number.



About 1600 A.D., Simon Stevin recognized that the decimal system could be used to express fractional measures by extending the place value relationship to the right of the units' position.

Thus, the digit immediately to the right of the units' position indicates tenths; the next to the right, hundredths; and so on.

Various methods of indicating the position of the units in a written number have been used. The most common practice is to place a dot called the **decimal point** in front of the digit in the tenths' position.

There are different ways to read decimal fractions. Since 0·23 is $\frac{2}{10} + \frac{3}{100}$ which is $\frac{23}{100}$, then 0·23 may be read twenty-three hundredths. Similarly, 0·257 is 257 thousandths; 0·2875 is 2,875 ten-thousandths; 0·056 is 56 thousandths, etc. Notice that the "name" of the decimal fraction is obtained from the place value of the digit on the extreme right. The number 26·534 is sometimes read twenty-six point five, three, four.

Example 1: Express as common fractions: (*a*) 0·05, (*b*) 3·125.

Solution: (*a*) $0{\cdot}05 = \frac{5}{100}$ (*b*) $3{\cdot}125 = 3\frac{125}{1000}$

$= \frac{1}{20}$. $= 3\frac{1}{8}$.

Example 2: Express these fractions in decimal notation:

(*a*) $5\frac{3}{10}$, (*b*) $\frac{7}{100}$.

Solution: (*a*) $5\frac{3}{10} = 5{\cdot}3$, (*b*) $\frac{7}{100} = 0{\cdot}07$.

Exercise

A

1. Read the following numbers:

0·2; 0·03; 0·165; 3·5; 5·87; 3·05; 256·890; 3·1416; 0·7; 0·005.

2. Write each of the following numbers using decimal notation:

(*a*) five tenths;
(*b*) thirty-five thousandths;
(*c*) five and seventy-five hundredths;
(*d*) six thousandths;
(*e*) thirty thousandths.

3. Express as equivalent numbers using decimal notation:

(*a*) $\frac{3}{10}$,
(*b*) $\frac{27}{100}$,
(*c*) $\frac{129}{1000}$,
(*d*) $\frac{5}{100}$,
(*e*) $\frac{7}{1000}$,
(*f*) $\frac{37}{10}$,
(*g*) $2\frac{9}{100}$,
(*h*) $5\frac{17}{1000}$.

4. Express as common fractions in their lowest terms:

(*a*) 0·5,
(*b*) 0·135,
(*c*) 0·0625,
(*d*) 0·125,
(*e*) 0·25,
(*f*) 0·375,
(*g*) 0·75,
(*h*) 0·05,
(*i*) 0·625,
(*j*) 0·875,
(*k*) 0·004,
(*l*) 0·001.

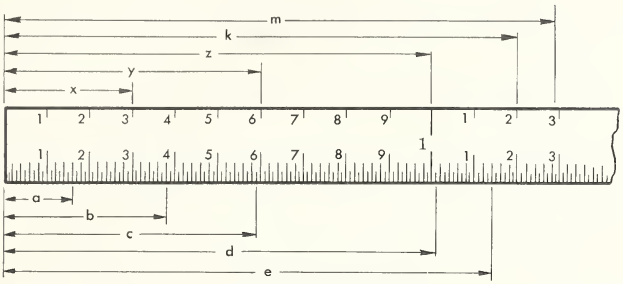
B

5. Express as common fractions in their lowest terms:

(*a*) 0·0625,
(*b*) 0·025,
(*c*) 0·1875,
(*d*) 0·96875,
(*e*) 0·515625,
(*f*) 0·984375,
(*g*) 0·000125,
(*h*) 0·00875.

Scale Reading

Below is a large scale illustration of 1·3 inches of a rule used by a large industrial firm which has adopted the decimal system for expressing all measures.



Using the upper side of the rule it is possible to read measures correct to the nearest tenth.

For measures less than 1 inch, a zero should be placed in the units' position to avoid any possibility of doubt. Here x is 0·3. Read the measures indicated by x, y, z, k and m.

The measure m is nearer to 1·3 than 1·2. If a more accurate reading for this is required, a scale divided into smaller units is necessary.

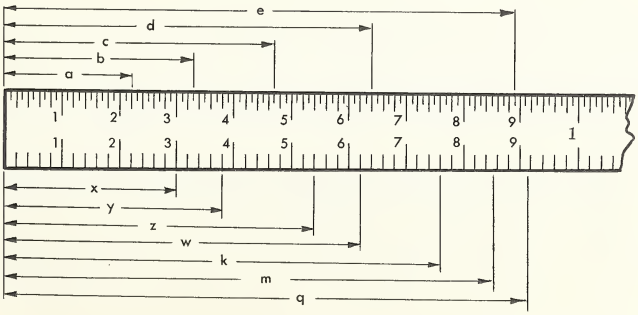
The smallest scale division represented on the lower side of this rule is $\frac{1}{100}$ or 0·01. This scale enables us to read to the nearest hundredth.

Read the measures indicated by a, b, c, d and e.

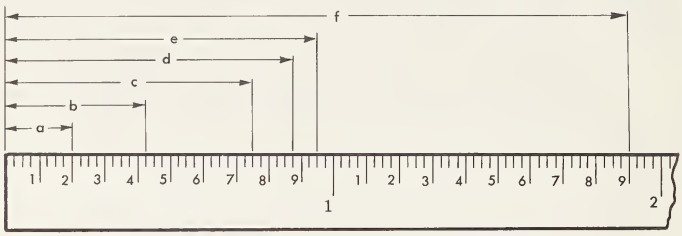
Exercise

A

1. Study the scale shown below. State the measures indicated by each of the letters in the diagram.



2. The diagram below represents a scale commonly used. The largest division represents one inch.



(*a*) What measure is represented by each of the larger divisions into which the inch is divided?

(*b*) Express in decimal notation the measure represented by the smallest division.

(*c*) State the measures represented by each of the letters *a*, *b*, *c*, *d*, *e* and *f* respectively.

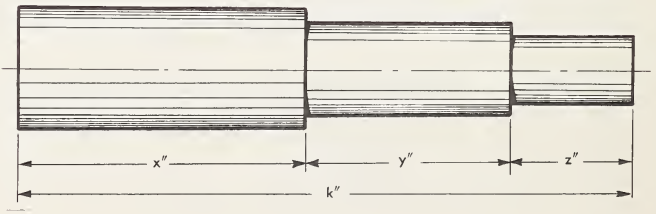
3. When carrying out additions or subtractions with decimal numbers, you write the numbers so that the decimal points are directly under each other. Explain why this is done.

B

4. A casting in the rough weighs 25·7 lb. and when finished weighs 17·95 lb. What weight of metal was removed?

5. A piece of flat work is $3\frac{1}{2}''$ thick. It is required to be 3·15″ thick. What depth of cut must be taken to reduce it to the required thickness?

6. In the drawing, $x = 3{\cdot}875$, $y = 2{\cdot}682$ and $z = 1{\cdot}893$.



(*a*) Find the measure k of the over-all length.

(*b*) How much greater is x than y?

(*c*) How much must be added to z to make it equal to y?

(*d*) How much greater is the sum of y and z than x?

7. The small diameter of a piece of tapered work is 1·598″ and the large diameter is 1·875″. What is the difference in the measures of the diameters?

8. From the sum of 2·56, 56·28 and 33·025 subtract the sum of 5·875 and 24·863.

9. (*a*) Subtract 0·387 from 1.

(*b*) How much must be added to 0·035 to make it equal 1?

10. Find the sum when 3·875 is taken as an addend 4 times.

11. From 1·5 subtract 0·375 four times successively.

12. How many times may 0·875 be subtracted successively from 2·625?

C

13. On a certain inspection trip four stations A, B, C and D, in that order, were visited. The automobile speedometer showed the following readings in miles: starting at station A the reading was 2563·8; at B 2578·4; at C 2596·7; at D 2603·9 and on return to A 2619·3. From the recorded data calculate the following:

(i) the distance between successive stations;

(ii) the distance travelled from A by way of B to get to each of the stations;

(iii) the total distance travelled;

(iv) the shortest distance from A to each station, assuming there are no other roads.

14. A family budget allows $25 a week for purchases to be made at a chain grocery store. The receipt slips are filed each time purchases are made. For the first week the totals were, $0·73, $8·01, $11·02, $3·58. The second week the totals were $5·98, $4·76, $15·92. The third week the totals were $0·78, $3·47, $2·18, $8·17, $9·83. The fourth week the totals were $2·89, $9·87 and $7·98.

(*a*) Find the total spent each week. State how much each total was under or over the weekly budget.

(*b*) Find the total for the four weeks and compare it with the suggested budget.

15. Gauge blocks are small rectangular blocks of steel which are finished to an accuracy of a few millionths part of an inch of their specified size. A set consists of 4 series of blocks.

Courtesy Pratt & Whitney

Precision Gauge Blocks

(*a*) In the first series there are 9 blocks ranging in size from 0·1001″ to 0·1009″. Write down the measure of the thickness of each block in this set.

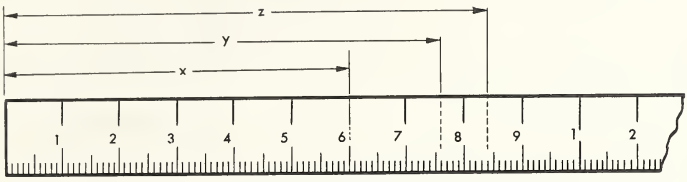
(*b*) The second series consists of 49 blocks beginning with a thickness of 0·101″ and increasing in size by 0·001″. Write down the sizes of the blocks in the second series.

(*c*) The third series consists of 19 blocks beginning with one 0·050″ and increasing in size by 0·050″. Write down the sizes of the blocks in the third series.

(*d*) The fourth series is a set of blocks of which the thicknesses are 1·000″, 2·000″, 3·000″ and 4·000″.

Exact and Approximate Measures

The cost of a certain suit of clothes is \$65. Sixty-five indicates **exactly the cost**. When we say, however, that we weigh 163 lb., 163 **approximates** the measure of our weight. The actual weight may be a little more or a little less than 163 lb. It should mean, however, that our weight is between 162·5 lb. and 163·5 lb. *Numbers obtained by counting usually give the measure exactly; those obtained by using measuring instruments always approximate the actual measure.*



On the upper scale illustrated, x represents 0·6 or $\frac{3}{5}$. If the scale were "exact", which is hardly possible, 0·6 would be an exact measure. If a line "exactly" 3 inches long were divided "exactly" into 5 equal parts each part would be "exactly" $\frac{3}{5}$ inches or 0·6 inches long. Actual measures, however, are more like those illustrated by y and z; y is nearer to 0·8 than it is to 0·7; z is nearer to 0·8 than 0·9. Using the upper scale, each of these measures would be indicated by 0·8. This means that both lie somewhere between 0·75 and 0·85.

From this scale the measures can be read to the nearest tenth of a unit. One-tenth or one decimal place is the **precision** of this measure. If more accurate measures were required for x, y and z we could use a scale divided into smaller fractional parts as illustrated on the lower part of the diagram.

From the lower scale the measures can be read to the nearest hundredth of a unit and they can be expressed to greater precision; namely, to hundredths or to two places of decimals. Note, also, that from this scale the measure z becomes slightly different from the measure y. The readings from the upper scale did not distinguish between these two lengths.

In actual measurements errors may arise from causes other than the reading of the scale. In this elementary treatment these other errors are overlooked entirely.

Perhaps at first consideration our measuring does not appear to be very accurate, but we can agree to take a definite meaning from our numbers. It is usual to consider that the limits within which the actual measures are obtained are as follows:

(*a*) take one-half the *place* value of the last reliable digit and add to find the upper limit;

(*b*) subtract this to obtain the lower limit.

A measure expressed as 9·6 lies between 9·6 + 0·05 and 9·6 — 0·05 or between 9·65 and 9·55.

Later you will have occasion to use instruments by means of which very accurate measurements can be made.

When numbers which approximate measures are added or subtracted, the sum or difference is not reliable to a higher degree of precision than the least precise of the numbers used.

To "round off" a measure, the last kept digit is determined as follows: *If the digit following it is 5 or more, this last kept digit is increased by 1; if the digit following is 4 or less, no change is made in the last kept digit.*

Example 1: Find the perimeter of the rectangle with length 9·45″ and width 0·857″.

Solution 1: Round off the measure of the width to the nearest hundredth and use 0·86″.

Perimeter in inches is $2(9{\cdot}45 + 0{\cdot}86) = 2 \times 10{\cdot}31$.

$= 20{\cdot}62$

Solution 2: Perimeter in inches is $2(9{\cdot}45 + 0{\cdot}857)$

$= 2 \times 10{\cdot}307$

$= 20{\cdot}614$

$= 20{\cdot}61$.

These results differ slightly. Calculations with numbers which approximate the measures are quite complex. However, two commonly used working rules are adequate for most practical purposes:

For adding or subtracting with numbers which approximate measures either

round off all measures to the same degree of precision as the least precise and add or subtract

or

calculate with the numbers as given and round off the final result to the same degree of precision as the least precise of the original data.

Example 2: Find the difference in length and width of the rectangle in *example 1.*

Solution 1: When measures corrected to the same precision are used the difference in inches is 9·45 — 0·86

$= 8{\cdot}59$

Solution 2:

Difference in inches is 9·45 — 0·857

$= 8{\cdot}593$

$= 8{\cdot}59$ when rounded to the number of digits of the least precise measure.

Exercise

A

Consider each of the following measures and, having in mind the probable method of obtaining them, classify them as exact or approximate. Give a reason for your classification. If a measure is approximate, indicate the probable limits within which the measure is reliable.

1. The price a boy paid for a suit of clothes was \$35.
2. The weight of a bag of sugar is 2 lb.
3. The population of Ontario is 5,504,000.
4. In a certain city, the number of votes cast in a recent civic election was 21,543.
5. A boy's age is 13 years 5 months.
6. The distance from Toronto to Montreal is 330 miles.
7. The width of a lot on which your house stands is 24′ 6″.
8. The number of teeth on a washing-machine gear is 48.
9. The window sash should be $32\frac{1}{4}$″ long and $24\frac{1}{2}$″ wide.
10. After I had driven five years, the speedometer reading was 65,506 miles.
11. The pay-roll for Jones Paving Co. for June 1959 was \$28,953·40.
12. The area of Canada as listed in "Canada and the United Nations" is 3,446,556 square miles.

13. (*a*) Which is the least precise of the following measures:
8·45, 9·00, 256·5, 3·687, 0·008, 0·0348?
(*b*) Which of the above measures has the greatest degree of precision?

14. Indicate the limits between which you would expect each of the following measures to be found:
(*a*) 3·5, (*b*) 0·456, (*c*) 0·2876, (*d*) 0·8350,
(*e*) 256, (*f*) 3·52, (*g*) 23·0, (*h*) 23,
(*i*) 2·56, (*j*) 2·5.

15. Express each of the following measures correct to 3 place of decimals:
(*a*) 2·6854, (*b*) 0·9877, (*c*) 0·0375,
(*d*) 0·0098, (*e*) 6·0005, (*f*) 3·0004.

16. Express each of the following correct (i) to the nearest hundredth (ii) to the nearest tenth:
(*a*) 2·563, (*b*) 0·105, (*c*) 1·246,
(*d*) 3·049, (*e*) 2·985, (*f*) 0·087.

B

17. Express each of the following measurements to the same degree of precision and find their sum.

(*a*) 3·542″, 25·86″, 0·875″;
(*b*) 5′ 3·5″, 27′ 7·25″, 81′ 3·66″;
(*c*) 9 yd. 2 ft. 8 in., 6 yd. 1 ft. 3 in., 4 ft. 8 in.

18. Find the sum of the following measurements and express the result to the same degree of precision as the least precise.

(*a*) 23·78 cm., 48·296 cm., 5·872 cm.;
(*b*) 5 lb. 7 oz., 18 lb. 3·5 oz., 4 lb. 12·7 oz.;
(*c*) 5·36 m., 75·2 cm., 3·8 m.

19. Find the difference between:

(*a*) 25′ 9·6″ and 33′ 5″;
(*b*) 10 yd. 2 ft. 7 in. and 15 yd.;
(*c*) 353·8 gm. and 583 gm.

Express the result to the degree of precision of the least precise measurement.

Multiplication of Decimal Fractions

Example 1: Find the product of 3·2 and 2·7.

Method 1:
$$3\cdot2 \times 2\cdot7 = \frac{32}{10} \times \frac{27}{10}$$
$$= \frac{32 \times 27}{100}$$
$$= \frac{864}{100}$$
$$= 8\tfrac{64}{100}$$
$$= 8\cdot64.$$

Method 2:
$$\begin{array}{r} 3\cdot2 \\ 2\cdot7 \\ \hline 224 \\ 64 \\ \hline 8\cdot64 \end{array}$$
$3\cdot2 \times 2\cdot7 = 8\cdot64.$

Example 2: Find the product of 21·3 and 1·84.

Method 3: Now 21·3 is close to 20 and 1·84 is close to 2. Use $2 \times 20 = 40$ as a trial product to fix the position of the decimal. Multiply 213×184 in the usual way and get 39192.

$$\therefore\ 21\cdot3 \times 1\cdot84 = 39\cdot192.$$

Exercise

A

1. State the product of:

(*a*) $0\cdot1 \times 0\cdot3$, (*b*) $0\cdot5 \times 0\cdot9$, (*c*) $1\cdot2 \times 1\cdot2$,
(*d*) $1\cdot1 \times 0\cdot8$, (*e*) $0\cdot5 \times 0\cdot5$, (*f*) $0\cdot3^2$,

(*g*) $0{\cdot}4^2$, (*h*) $0{\cdot}3^3$, (*i*) $0{\cdot}16 \times 0{\cdot}3$,
(*j*) $2{\cdot}4 \times 0{\cdot}5$, (*k*) $0{\cdot}02^2$, (*l*) $7 \times 0{\cdot}7$,
(*m*) $2{\cdot}0 \times 0{\cdot}002$, (*n*) $0{\cdot}003 \times 0{\cdot}04$, (*o*) $0{\cdot}13^2$,
(*p*) $0{\cdot}15^2$, (*q*) $0{\cdot}75 \times 16$, (*r*) $0{\cdot}25 \times 2{\cdot}8$,
(*s*) $0{\cdot}33 \times 0{\cdot}04$, (*t*) $120 \times 0{\cdot}125$.

2. State the square of each of the following measures:

(*a*) 1·1, (*b*) 1·2, (*c*) 1·9,
(*d*) 0·1, (*e*) 0·3, (*f*) 0·4,
(*g*) 0·9, (*h*) 0·01, (*i*) 0·03,
(*j*) 0·04.

3. Express each of the following measures in decimal form without using powers of 10:

(*a*) $2{\cdot}345 \times 10^2$, (*b*) $3{\cdot}87 \times 10^3$,
(*c*) $2{\cdot}71 \times 10^6$, (*d*) $12{\cdot}3875 \times 10^3$,
(*e*) $5{\cdot}1 \times 10^3$, (*f*) $3{\cdot}57 \times 0{\cdot}1$,
(*g*) $2{\cdot}96 \times 0{\cdot}001$, (*h*) $7{\cdot}83 \times \frac{1}{1000}$,
(*i*) $8{\cdot}53 \times \frac{1}{10^3}$.

4. Estimate each of the following products:

(*a*) $2{\cdot}15 \times 0{\cdot}506$, (*b*) $0{\cdot}256 \times 32{\cdot}7$, (*c*) $0{\cdot}05 \times 46{\cdot}3$,
(*d*) $124{\cdot}3 \times 9{\cdot}98$, (*e*) $389{\cdot}5 \times 19{\cdot}87$, (*f*) $0{\cdot}001 \times 56{\cdot}7$.

B

5. Find the product of each of the following:

(*a*) $2{\cdot}15 \times 24{\cdot}6$, (*b*) $356 \times 29{\cdot}2$,
(*c*) $0{\cdot}532 \times 9{\cdot}86$, (*d*) $5{\cdot}6^2$,
(*e*) $0{\cdot}83^3$, (*f*) $26{\cdot}5 \times 3{\cdot}4 \times 8{\cdot}53$.

6. A shipment of 250 sheets of galvanized iron No. 20 gauge is received. This is to be piled, one sheet above the other, in a stock room. What height will this pile be if each sheet of No. 20 gauge is 0·0359 inches in thickness?

7. Calculate the volume of a cube if the length of each side is 1·36 inches.

8. A steel drum contains 45 gallons of shellac. The drum weighs 35·5 lb. and a gallon of shellac approximately 7·85 lb. What is the total shipping weight of a drum of shellac?

9. Write down the cubes of each of the following:

(*a*) 1·2, (*b*) 2·5, (*c*) 0·1, (*d*) 0·2, (*e*) 0·4, (*f*) 0·9.

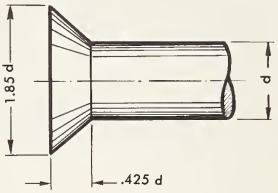
10. The length of a bearing for certain purposes is estimated to be 2·5 times the diameter of the shaft it is to support. Find the lengths of suitable bearings for shafts of which the diameter is

(*a*) 0·875″, (*b*) 1·250″, (*c*) 1·375″.

11. An order called for 75 steel pins each 1·375 in. long, to be cut from a steel bar. If $\frac{1}{8}''$ is allowed for cutting and finishing each pin, what would be the length of the bar used?

12. The outside diameter of a 6-inch copper pipe is 6·625 in. The wall thickness of "extra strong" piping of this size is 0·437 in. What is the actual inside diameter?

13. A standard rivet as shown on the diagram has the diameter across the head equal to 1·85 times the diameter of the rivet. The depth of the head is equal to 0·425 times the diameter of the rivet.



Complete the following table for various diameters of rivets.

Diameter of rivet in inches	*Diameter across top of rivet in inches*	*Depth of head of rivet in inches*
0·250		
0·375		
0·188		
0·125		

C

14. How many pounds of water are there in a tank which contains 805·7 cu. ft.? A cu. ft. of water weighs 62·32 lb.

15. Find the area of the circle when the radius is 7·5″.

16. If gasoline costs $46\frac{9}{10}$ cents per gallon, find the cost of 12·7 gallons.

Multiplication and Division with Approximate Data

In addition and subtraction with approximate data, the results were rounded off to the degree of *precision* of the least precise measure of the data.

To round off our results of multiplication or division we need to understand the terms **degree of accuracy** and **significant digits**.

To help to understand the meaning of the degree of accuracy, examine the two measures 1·25 and 0·06 which, of course, have the same degree of precision since the error in each is assumed to be less than 0·005.

The percentage error of the first, however, is $\frac{0{\cdot}005 \times 100\%}{1{\cdot}25} = 0{\cdot}4\%$ and the percentage error of the second is $\frac{0{\cdot}005}{0{\cdot}06} \times 100\% = 0{\cdot}8\%$. The first measure has *less* percentage error; therefore a *greater* degree of accuracy.

Again, let us compare the percentage error of two measures such as 2·75 and 0·0275.

The percentage error of 2·75 is $\frac{0{\cdot}005}{2{\cdot}75} \times 100\% = 0{\cdot}2\%$.

The percentage error of 0·0275 is $\frac{0{\cdot}00005}{0{\cdot}0275} \times 100\% = 0{\cdot}2\%$.

These two measures have the same degree of accuracy although their degrees of precision are quite different.

It will be noted that the two measures 0·0275 and 2·75 each have three significant digits and they have the same degree of accuracy. *In general, numbers with the same number of significant digits have approximately the same degree of accuracy.*

The measure 1·25 has 3 significant digits while 0·06 has 1 significant digit, the zero acting as a place holder only. The number 1·25 will have a greater degree of accuracy than 0·06. This agrees with the calculations of their percentage errors above.

To determine the number of significant digits in a number:

(i) the digits 1, 2, 3, etc. are always significant digits;
(ii) in a number like 0·003 the zeros are place holders only and are not significant digits;
(iii) zeros used between other digits like 4·003 are always significant;
(iv) zeros at the end of a number may or may not be significant.

In a measure like 30 inches, if it is known that this measure is to the nearest inch then the zero is significant. When the distance between two towns is given as 450 miles it is quite likely this will be given to the nearest 10 miles and then the zero is not significant. In the latter case, the number would be considered to have only two significant digits.

In practice, zeros at the end of an integer are not considered significant unless other information about the measure indicates that they are. When a measure is expressed as 20·0 the only reason for the zero after the decimal place is to indicate that the measure was made to the nearest tenth and here the two zeros are significant and this would be considered a number with 3 significant digits.

An example will illustrate the usual practice in rounding off results of multiplication.

Find the area of rectangle if the length is 2·3 inches and the width is 1·6 inches.

Area in sq. in. is $2{\cdot}3 \times 1{\cdot}6$
$= 3{\cdot}7.$

$$\begin{array}{r} 2\cdot③ \\ 1\cdot⑥ \\ \hline 1③⑧ \\ 2③ \\ \hline 3\cdot⑥⑧ \end{array}$$

In the multiplication shown on page 87, digits of doubtful accuracy are circled. The digit 8 in the hundredths position in the product 3·68 indicates that the area is nearer to 3·7 than 3·6. It seems reasonable to take 3·7 as the measure of the area.

This conclusion is further justified by the fact that 3·7 lies between the least and the greatest possible measures of the area, calculated by using the lower and upper limits, respectively, of the dimensions given for the rectangle. It will be noticed that the number of digits expressed in the final product is the same as the number of digits in the separate measures from which it was obtained.

A reasonable degree of accuracy in calculations involving multiplication or division can be obtained by observing the following rules:

(*a*) Wherever possible during the calculation, keep one more digit than will be kept in the final result;

(*b*) Round off the final result to not more than the number of digits in the least accurate of the given data.

In the example above the degree of accuracy of the calculation could be expressed in any one of three ways, namely, (*a*) *to 2 significant digits*, (*b*) *to 1 decimal place*, (*c*) *to the nearest tenth.*

The student is reminded that the procedure for multiplication or division with approximate data differs slightly from that for addition or subtraction.

This, of course, has been a very elementary treatment of operations with numbers which approximate measures but it will serve as a reasonably accurate guide in rounding off final results.

Exercise

A

1. State the number of significant digits in each of the following measures:

(*a*) 35, (*b*) 0·035, (*c*) 35·0, (*d*) 350,
(*e*) 0·305, (*f*) 350·0, (*g*) 0·87, (*h*) 0·8300.

2. State the degree of accuracy of each of the following measures:

(*a*) 235, (*b*) 2·750, (*c*) 0·483,
(*d*) 0·0005, (*e*) 37,658, (*f*) 38·4.

3. Express each of the following corrected to 3 significant digits:

(*a*) 56·38, (*b*) 0·05687, (*c*) 389,270,
(*d*) 4·568, (*e*) 0·1248, (*f*) 3,456,000.

B

4. The dimensions of a rectangle are found to be 0·23″ and 1·6″ respectively.

(*a*) Assuming the usual manner for interpreting measured data, indicate the smallest possible dimensions;

(*b*) Using the results of part (*a*), find the smallest possible area for this rectangle;

(*c*) What are the largest possible dimensions?
(*d*) What is the largest possible area?
(*e*) Show that the area found by using the procedure for multiplication with approximate data lies between the limits indicated by the results in (*b*) and (*d*).

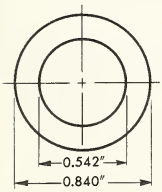
5. Multiply 3·142 by 6·854 and express the result to 3 significant digits.

6. The length of a rectangular steel key used to secure a pulley on a shaft is given as 2·50 inches. The dimensions of its end are 0·350 inches and 0·255 inches respectively.
(*a*) What is the area of the end of the key?
(*b*) What is the volume of the key?
(*c*) If steel weighs 0·2835 lb. per cu. in., calculate its weight.

Division of Decimal Fractions

Fractions expressed in decimal notation are simply an extension of the notation used for integers. We have seen that the operations of addition, subtraction and multiplication are carried out in a manner similar to those for integers. Similarly when decimal fractions are involved in division, the operation is carried out as for integers; some additional care, however, must be exercised to determine the position of the decimal point.

Division by Integers



Example 1: The inside and outside diameters of "extra strong" copper pipe, illustrated in a catalogue, are given as 0·840″ and 0·542″. What is the thickness of the material in the walls?

Solution: The difference in the diameter in inches is $0{\cdot}840 - 0{\cdot}542$

$$= 0{\cdot}298$$

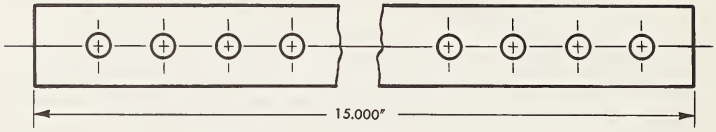
The thickness of the material in inches is $\frac{0{\cdot}298}{2}$

$$= 0{\cdot}149.$$

Example 2: Twelve holes are to be equally spaced along a bar 15·000″ long. Calculate the distance between the centres if the distance from the end of the bar to the centre of the end hole is equal to the distance between centres. Assume that the measurements are made to the nearest thousandth of an inch.

Solution:

To obtain centres for 12 holes 13 spaces are required.



$\therefore$ the distance between centres in inches is $\dfrac{15{\cdot}000}{13}$

$= 1{\cdot}1538$

The distance between centres to the nearest thousandth of an inch is 1·154.

```
      1·1538
13)15·0000
   13
   ---
    20
    13
    --
     70
     65
     --
      50
      39
      --
      110
      104
      ---
        6
```

Division by Non-integral Divisors

Division enables us to calculate the number of times one quantity is contained in another if each is expressed in terms of the same unit.

Consider the situation in the following problem:

A laminated pole for a motor is to be built from laminations each 0·035 inches thick. How many will be required to construct a pole as nearly 4 inches in length as possible?

Solution: The number of laminations is $\dfrac{4}{0{\cdot}035}$.

In this division the divisor is not an integer. Considering the quotient as a fraction, we may change the denominator to an integer by multiplying it by 1000 and by multiplying the numerator by 1000 we keep the value of the fraction the same.

Thus $4 \div 0{\cdot}035 = \dfrac{4}{0{\cdot}035}$

$= \dfrac{4000{\cdot}}{35{\cdot}}$

$= 114{\cdot}2$

The number of laminations is 114.

```
      114·2
35)4000·
   35
   --
    50
    35
    --
    150
    140
    ---
     100
      70
      --
      30
```

Example 1: Divide 7·80 by 0·964.

Solution: $7{\cdot}80 \div 0{\cdot}964$

$$= \frac{7{\cdot}80}{0{\cdot}964}$$

$$= \frac{7{\cdot}80 \times 1000}{0{\cdot}964 \times 1000}$$

$$= \frac{7800{\cdot}}{964{\cdot}}$$

$$= 8{\cdot}09$$

$$\begin{array}{r} 8.091 \\ 964\overline{)7800\cdot} \\ 7712 \\ \hline 8800 \\ 8676 \\ \hline 1240 \\ 964 \\ \hline \end{array}$$

Check: $7{\cdot}80 \div {\cdot}964$ is approximately $8 \div 1 = 8$.

Example 2: Divide 0·164 by 3·14.

Solution: $0{\cdot}164 \div 3{\cdot}14 = \frac{0{\cdot}164}{3{\cdot}14}$

$$= \frac{16{\cdot}4}{314{\cdot}}$$

$$= 0{\cdot}052$$

$$\begin{array}{r} {\cdot}0522 \\ 314\overline{)16{\cdot}40} \\ 15\,70 \\ \hline 700 \\ 628 \\ \hline 720 \\ 628 \\ \hline 92 \end{array}$$

Check: $0{\cdot}164 \div 3{\cdot}14$ is approximately $0{\cdot}2 \div 3 = 0{\cdot}06$.

Exercise

A

1. Simplify:

(*a*) $\frac{0{\cdot}4}{2}$, (*b*) $\frac{0{\cdot}5}{2}$, (*c*) $\frac{2{\cdot}56}{10}$, (*d*) $\frac{0{\cdot}64}{8}$,

(*e*) $\frac{0{\cdot}024}{12}$, (*f*) $\frac{1{\cdot}7}{100}$, (*g*) $\frac{51{\cdot}2}{8}$, (*h*) $\frac{0{\cdot}001}{4}$.

2. Simplify:

(*a*) $\frac{0{\cdot}04}{0{\cdot}2}$, (*b*) $\frac{1{\cdot}8}{0{\cdot}03}$, (*c*) $\frac{0{\cdot}01}{0{\cdot}1}$, (*d*) $\frac{27{\cdot}}{0{\cdot}09}$,

(*e*) $\frac{1{\cdot}}{0{\cdot}4}$, (*f*) $\frac{0{\cdot}5}{0{\cdot}8}$, (*g*) $\frac{3{\cdot}5}{0{\cdot}05}$, (*h*) $\frac{0{\cdot}72}{0{\cdot}8}$.

3. Noting that $364 \div 26 = 14$ express each of the following quotients in decimal notation:

(*a*) $36{\cdot}4 \div 26$, (*b*) $0{\cdot}364 \div 26$,

(*c*) $0{\cdot}0364 \div 26$, (*d*) $364 \div 2{\cdot}6$,

(*e*) $364 \div 0{\cdot}26$, (*f*) $364 \div 0{\cdot}026$.

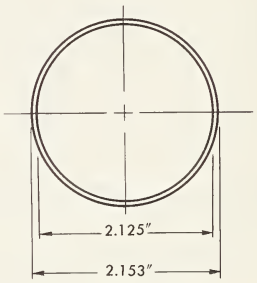
4. Express the following quotients in decimal form:

(*a*) $10 \div 100$, (*b*) $2{\cdot}1 \div 7$,
(*c*) $0{\cdot}24 \div 3$, (*d*) $2{\cdot}5 \div 25$,
(*e*) $3{\cdot}5 \div 0{\cdot}7$, (*f*) $0{\cdot}27 \div 30$,
(*g*) $0{\cdot}0042 \div 0{\cdot}006$, (*h*) $45 \div 0{\cdot}009$.

5. As a matter of convenience, composing-room time is stated as a decimal fraction of an hour instead of hours and minutes.

(*a*) If a man works 1 hour and 30 minutes how would his time be stated?
(*b*) A printer worked 3 hours and 15 minutes. How would this be stated?
(*c*) If a man worked 1 hour and 36 minutes how would this be expressed?

6. Find the thickness of the metal in the diagram at the right.



7. If sheets of paper 0·036 inches thick were piled on top of each other how many would be in a pile 6·00 inches thick?

8. In order to obtain as accurate a measure as possible for the diameter of a ball-bearing, readings were taken with a ten-thousandth micrometer. The measures thus found were: 0·3249, 0·3251, 0·3252, 0·3248, 0·3246, 0·3253. Find the average of these readings correct to the nearest ten thousandth.

9. One cubic foot of aluminum weighs 164 lb. What would 1 cu. in. of aluminum weigh correct to the nearest thousandth?

10. What would be the unit cost of 1,000 bolts which cost $4·50?

11. In a sheet-metal stock-room is a quantity of No. 31 gauge galvanized iron, thickness 0·0105 inches. How many sheets would be in a pile 5 feet high?

12. The diameter of a piece of stock in a lathe is 1·564 inches and it is required to be 1·560 inches. How deep a cut should be taken to reduce the diameter to this size?

13. An American gallon contains 231·0 cu. in. and a Canadian or Imperial gallon contains 277·4 cu. in. If a gallon of gasoline costs 42 cents in Canada what would be the comparable cost of an American gallon?

C

14. In parallel circuits in electricity, calculations similar to the following occur. Since 1 is exact, simplify these expressions to the accuracy indicated by the denominators.

(*a*) $\frac{1}{2{\cdot}1} + \frac{1}{2{\cdot}5}$; (*b*) $\frac{1}{65{\cdot}3} + \frac{1}{48{\cdot}1} + \frac{1}{23{\cdot}6}$.

15. The measure of the resistance in ohms of an electrical circuit is obtained by dividing the voltage of the circuit in volts by the measure of the current in the circuit in amperes. Find the resistance in ohms in the following circuits:

(*a*) voltage is 60·0 volts and the current is 1·5 amperes;
(*b*) voltage is 110·3 volts and the current is 6·8 amperes;
(*c*) voltage is 115 volts and the current is 2·25 amperes.

16. How many feet of No. 0 copper wire would be required to provide a resistance of 1·500 ohms if this wire has a resistance of 0·1003 ohms per 1,000 feet?

17. The smallest rubber-covered copper wire used in house wiring has a resistance of 2·52 ohms per 1,000′. What is the resistance of

(*a*) $\frac{1}{2}$ mile of this wire; (*b*) a piece 200′ long?

Converting Common Fractions to Decimal Fractions

In the home, in business, in the shop, in the laboratory or wherever measures are used it is often necessary to convert a measure expressed in one notation to an equivalent measure in the other.

Example 1: Find the decimal equivalent of $\frac{7}{40}$.

Solution: $\frac{7}{40} = 7 \div 40$

$= 0{\cdot}175.$

$$\begin{array}{r} 0{\cdot}175 \\ 40\overline{)7{\cdot}000} \\ 4\,0 \\ \hline 3\,00 \\ 2\,80 \\ \hline 200 \\ 200 \\ \hline \end{array}$$

Actually there are few fractions which can be expressed exactly in the ordinary decimal notation. All common fractions, however, can be expressed in decimal notation to any desired degree of approximation.

Example 2: Express $\frac{5}{7}$ as a decimal fraction correct to three places of decimals.

Solution: $\frac{5}{7} = 5 \div 7$

$= 0{\cdot}7142$

$= 0{\cdot}714$ correct to 3 places of decimals.

$$\begin{array}{r} 0{\cdot}7142 \\ 7\overline{)5{\cdot}0000} \end{array}$$

Example 3: Express $\frac{7}{9}$ as a decimal correct to 4 significant digits.

Solution: $\frac{7}{9} = 7 \div 9$

$= 0{\cdot}77777\cdots$ 0·777777...

$= 0{\cdot}7778$ correct to 4 significant digits. $9\overline{)7{\cdot}0000}$

When $\frac{7}{9}$ is expressed as a decimal the number consists of 7 repeated indefinitely. Such a decimal fraction is called a *repeating decimal* and it may be written $0{\cdot}\dot{7}$; the dot over the digit indicates that it repeats itself.
Thus, $0{\cdot}\dot{7}$ expresses exactly the decimal equivalent of $\frac{7}{9}$.

Example 4: Express 0·225 as a common fraction in its lowest terms.

Solution: $0{\cdot}225 = \frac{225}{1000}$ Divide numerator and denominator by 25.

$= \frac{9}{40}$.

Example 5: Change the ruler fraction $\frac{5}{16}$ to a decimal fraction.

Method 1: We know that $\frac{1}{8} = 0{\cdot}125$

Therefore $\frac{1}{16} = \frac{1}{2}$ of $0{\cdot}125$

$= 0{\cdot}0625$

$\therefore \quad \frac{5}{16} = 5 \times 0{\cdot}0625$

$= 0{\cdot}3125.$

or *Method 2:*

$\frac{5}{16} = \frac{1}{4} + \frac{1}{16}$

$= 0{\cdot}25 + 0{\cdot}0625$

$= 0{\cdot}3125.$

Exercise

A

1. Express as decimal fractions:

(*a*) $\frac{1}{2}$, (*b*) $\frac{1}{4}$, (*c*) $\frac{1}{8}$, (*d*) $\frac{3}{4}$,
(*e*) $\frac{3}{8}$, (*f*) $\frac{5}{8}$, (*g*) $\frac{7}{8}$, (*h*) $\frac{7}{10}$,
(*i*) $\frac{1}{20}$, (*j*) $\frac{9}{20}$, (*k*) $\frac{1}{3}$, (*l*) $\frac{2}{3}$,
(*m*) $\frac{3}{5}$, (*n*) $\frac{1}{7}$, (*o*) $\frac{1}{9}$.

2. Express as common fractions:

(*a*) 0·8, (*b*) 0·3, (*c*) 0·125, (*d*) 0·06,
(*e*) 0·33, (*f*) 0·007, (*g*) 0·55, (*h*) 0·002,
(*i*) 0·67, (*j*) 0·875.

B

3. Express as decimal fractions correct to 3 significant digits each of the following:

(*a*) proper fractions in which the denominator is 7,
(*b*) proper fractions in which the denominator is 9,
(*c*) proper fractions in which the denominator is 11,
(*d*) proper fractions in which the denominator is 12.

4. Complete a table of equivalents for ruler fractions as suggested below.

DECIMAL EQUIVALENT TABLE

$\frac{1}{2} = 0{\cdot}5$	$\frac{1}{4} = 0{\cdot}25$	$\frac{1}{8} = 0{\cdot}125$	$\frac{1}{16} = \frac{1}{2}$ of $0{\cdot}125 = 0.0625$
	$\frac{3}{4} = 0{\cdot}75$	$\frac{3}{8} = 3 \times 0{\cdot}125 = 0{\cdot}375$	$\frac{3}{16} = 3 \times 0{\cdot}0625 = 0{\cdot}1875$
		$\frac{5}{8}$	$\frac{5}{16}$
		$\frac{7}{8}$	$\frac{7}{16}$
			etc.

5. Express exactly in decimal fraction notation all proper fractions in which the denominators are 20.

6. Express exactly using the repeating decimal notation:

(*a*) $\frac{1}{3}$, (*b*) $\frac{2}{3}$, (*c*) $\frac{5}{6}$, (*d*) $\frac{2}{9}$, (*e*) $\frac{1}{11}$.

7. Express exactly each of the following measures as equivalent measures using common fraction notation:

(*a*) 0·125, (*b*) 0·175,
(*c*) 0·0825, (*d*) 0·025,
(*e*) 0·00625, (*f*) 0·04375,
(*g*) 0·09375, (*h*) 0·08125,
(*i*) 0·9750.

8. Using the table of decimal equivalents constructed in No. 4, find the nearest ruler fraction for each of these decimal fractions:

(*a*) 0·34375, (*b*) 0·688,
(*c*) 0·810, (*d*) 0·295,
(*e*) 0·175, (*f*) 0·048,
(*g*) 0·511, (*h*) 0·998.

9. State which number in each of the following pairs is the larger:

(*a*) $\frac{3}{4}$ or 0·749, (*b*) $\frac{7}{16}$ or 0·428,
(*c*) $3\frac{1}{7}$ or 3·1416, (*d*) $\frac{3}{16}$ or 0·1895,
(*e*) $\frac{3}{7}$ or 0·438, (*f*) $\frac{17}{32}$ or 0·5305.

C

10. Express exactly in decimal fraction notation all proper fractions in which the denominators are 40.

11. Express exactly in decimal fraction notation all proper fractions (except those equivalent fractions included above) in which the denominators are 80.

12. By first changing to decimal fractions arrange the following in order of magnitude:

(*a*) $\frac{7}{40}$, $\frac{5}{32}$, $\frac{3}{14}$; (*b*) $\frac{15}{21}$, $\frac{7}{10}$, $\frac{9}{15}$;
(*c*) $\frac{15}{24}$, $\frac{7}{10}$, $\frac{9}{13}$; (*d*) $2\frac{7}{12}$, $2\frac{8}{15}$, $2\frac{59}{100}$.

Review Exercise

A

1. Read the following numbers:

(*a*) 5,364,500, (*b*) 30·03, (*c*) 0·002, (*d*) 0·0090,
(*e*) 72·3, (*f*) 3,000,000,000, (*g*) 0·000008.

2. By approximating the product place the decimal point in the following products and quotients which contain all the necessary significant digits after the calculation has been completed.

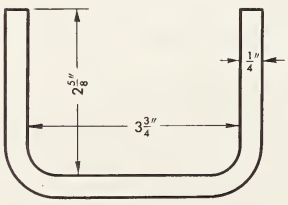
Product expressed	*Significant digits of the product*	*Quotient expressed*	*Significant digits of the quotient*
$1000 \times 0{\cdot}03$	30	$275 \div 1100$	25
$4{\cdot}2 \times 2{\cdot}1$	88	$16 \div 25$	64
$0{\cdot}04 \times 0{\cdot}03$	12	$1600 \div 25$	64
$3{\cdot}1416 \times 0{\cdot}875$	275	$0{\cdot}03 \div 0{\cdot}5$	6
$0{\cdot}2 \times 0{\cdot}2$	4	$0{\cdot}464 \div 0{\cdot}016$	29
$0{\cdot}081 \times 0{\cdot}064$	52	$0{\cdot}780 \div 0{\cdot}964$	809

B

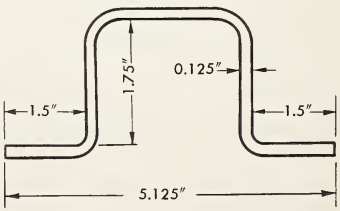
3. In Canada the law provides that 1 pound avoirdupois contains 7,000 grains. How many grains are in an ounce avoirdupois?

4. Letters for delivery in Canada, U.S., or the United Kingdom cost 5 cents for the first ounce and 3 cents for each additional ounce or fraction thereof. If a typed manuscript weighs 8 oz. what will it cost to send it letter rate?

5. Find the length of the stock needed to bend into the following shapes; allow $\frac{1}{2}$ of the thickness of the stock extra for each bend:



(*a*)

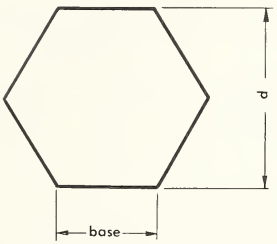


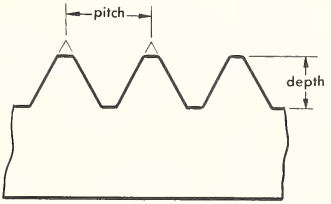
(*b*)

6. A steel bar 0·625″ in diameter weighs 1·043 lb. per foot. Find the weight of a tie of the same diameter and material which is 7·75 feet long.

7. A flat sheet of aluminum is 3′ by 2′ by 0·125″ thick. Aluminum weighs approximately 0·09 lb. per cu. in. Find the weight of this sheet. How much heavier is a similar sheet of brass which weighs 0·30 lb. per cu. in.?

8. The distance, d, across the flats of a hexagon is 1·732 times the length of the base. Find the distance across the flats for hexagons if the length of the base in each case is:
   (a) 1·25″, (b) 3·5″, (c) 0·75″.





9. The depth of a National Thread is 0·6495 times the pitch. Find the depth of each of these threads, to the nearest thousandth:
   (a) a 1″ diameter National Thread with a pitch of 0·125″;
   (b) a $\frac{3}{8}$″ diameter National Thread with a pitch of 0·0625″;
   (c) a $\frac{1}{4}$″ diameter National Thread with a pitch of $\frac{1}{20}$″.

10. A salesman sold \$1,250 worth of goods on a commission of $9\frac{1}{2}\%$. How much did he receive for his services?

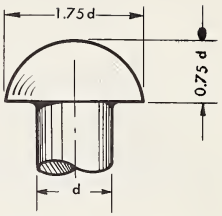
11. A contractor estimated that a pile of lumber contained 3,000 bd. ft. On measuring it he found it to contain 2,800 bd. ft. Express his error as a fraction of the correct measurement.

12. (a) How many days are there from January 1 to May 31 inclusive, in an ordinary year?
   (b) In an ordinary year what will be the date of the 250th day?

13. Find the daily pay of each of the following men who worked $7\frac{1}{2}$ hours each day:
   (a) a carpenter at \$2·25 per hour,
   (b) a painter at \$1·95 per hour,
   (c) a plumber at \$2·95 per hour,
   (d) an apprentice at \$1·37 per hour.

C

14. The diameter of a rivet head is 1·75 times the diameter of the rivet and the depth of the head is 0·75 times the diameter of the rivet.



Calculate these dimensions correct to 2 places of decimals for these rivets.

Diameter of rivet in inches	*Diameter of head of rivet in inches*		*Depth of head of rivet in inches*	
	Expressed	*Calculated*	*Expressed*	*Calculated*
0·25				
$\frac{3}{16}$				
$\frac{5}{32}$				

15. A year ago a man purchased an automobile for \$3,000. Make an estimate of his cost per mile using the following information:

(*a*) His year's mileage was 15,000.

(1) His average gasoline consumption was 20 miles per gallon and the average cost of gasoline was 41 cents per gallon.

(2) Every 1,000 miles he had the car serviced (greased, oil change etc.) at a cost of \$3·50 each time.

(3) Other expenses including licence, insurance etc. amounted to \$95.

(4) He paid his neighbour \$6 per month for the rental of a garage.

(5) He could have purchased bonds with \$3,000 and received interest at the rate of 3% per annum.

(6) He could, after 1 year, sell his car for \$2,400.

(*b*) If his year's mileage had been 10,000, what would have been the cost per mile?

CHAPTER 5

Percentage

Comparisons Expressed as Per Cent

For assisting at a grocery store after school, John receives 42 cents per hour. His friend Bill, who delivers at a drug store, receives 50 cents per hour.

We might compare their wages by stating that John's rate of pay is $\frac{42}{50}$ of Bill's. The two quantities are now compared by division.

By using the fundamental principle of fractions, we may express the measure $\frac{42}{50}$ as $\frac{84}{100}$. Then John's rate of pay is $\frac{84}{100}$ of Bill's, or John's rate of pay is 84 per cent of Bill's.

This means that if the boys work an equal number of hours, then for every 100 cents that Bill receives, John will receive 84 cents.

The symbol % is commonly used instead of writing the words per cent.

Example: A ball team won 6 out of 8 games played. What per cent of the games played were won?

Solution 1:

Fraction of the games won is $\frac{6}{8} = \frac{3}{4} = \frac{75}{100}$.
Per cent of games won is 75%.

Solution 2:

Per cent of the games won is $\frac{6}{8} \times 100\%$
$= 75\%$.

Rates Expressed as Per Cent

Example: A boy who worked part time in a sports shop purchased a bicycle for \$50. If the bicycle may be purchased regularly for \$60, what was the rate of discount his employer allowed him?

Solution:

Discount in dollars is $60 - 50 = 10$.
Rate of discount is $\frac{10}{60} \times 100\%$
$= 16\frac{2}{3}\%$.

Exercise

A

1. What per cent of 8 is 4?

2. What per cent of 60 is 15?

3. What per cent of 28 is 21?
4. What per cent of 120 lb. is 24 lb.?
5. What per cent of 400 people is 80 people?
6. What per cent of \$2·00 is \$1·50?
7. What per cent of 16 games is 5 games?
8. What per cent of 24 is
   (*a*) 6, (*b*) 12, (*c*) 16, (*d*) 18, (*e*) 48?
9. Express the following fractions as per cents:
   (*a*) $\frac{3}{4}$, $\frac{2}{3}$, $\frac{4}{5}$, $\frac{7}{8}$, $\frac{7}{10}$, $\frac{3}{20}$;
   (*b*) 0·5, 0·05, 0·005, 0·12, 0·015, 0·115;
   (*c*) $1\frac{1}{2}$, $1\frac{1}{4}$, $3\frac{3}{5}$, $4\frac{9}{10}$;
   (*d*) 1·8, 2·3, 3·45, 1·05, 1·07;
   (*e*) $\frac{2}{5}$, $\frac{3}{5}$, $\frac{7}{10}$, $\frac{3}{20}$, $\frac{1}{200}$, $\frac{3}{200}$, $\frac{1}{800}$.

B

10. Make a table similar to the one below. Calculate and fill in the information required to complete the table.

Common fraction	*Decimal fraction*	*Per cent*	*Common fraction*	*Decimal fraction*	*Per cent*
$\frac{1}{100}$			$\frac{7}{100}$		
$\frac{1}{50}$			$\frac{2}{25}$		
$\frac{3}{100}$			$\frac{9}{100}$		
$\frac{1}{25}$			$\frac{1}{10}$		
$\frac{1}{20}$			$\frac{1}{9}$		
$\frac{3}{50}$			$\frac{1}{8}$		
$\frac{1}{5}$			$\frac{1}{7}$		
$\frac{1}{4}$			$\frac{1}{6}$		
$\frac{1}{3}$			$\frac{3}{2}$		
$\frac{1}{2}$			2		
1					

11. What per cent is 4 of 20; of 60; of 120?
12. What per cent is 15 of 60; of 75; of 180?
13. What per cent of 90 is 18; of 40 is 80?
14. What per cent of 120 lb. is 18 lb.?
15. For an eight-hour day a man receives \$9·60. If he receives a raise of 12 cents per hour what per cent increase did he receive?

16. The number of students attending a school is 1,250. If 875 of these went to a rugby game what per cent of the enrolment attended the game?

17. In one day the inspection department of a certain firm rejected 24 castings because they were not up to standard. If a total of 260 were produced, what per cent were rejected?

18. A student was absent from school 3 days in a month which had 19 school days. What per cent of the time was he absent?

19. A student received 23 marks on a test for which the total marks were 30. What per cent of the total was his score?

20. (*a*) What per cent of 8 is 5?
(*b*) What per cent of 7 is 9?
(*c*) What per cent of 0·36 is 0·02?

21. (*a*) 56 is what per cent of 98?
(*b*) 35 is what per cent of 40?
(*c*) 0·3 is what per cent of 0·45?

22. (*a*) Express 35 as a per cent of 90.
(*b*) Express 30 as a per cent of 20.
(*c*) Express 8 as a per cent of 2.
(*d*) Express 375 as a per cent of 700.

23. For a large school the Board of Education paid the following bills for maintenance:

coal	$71,000
cleaning	$17,500
painting	$ 2,500
repairs	$28,500
miscellaneous	$ 6,500

Find the per cent that each of these is of the total bill.

24. Two years ago a motor car company produced 52,400 cars. Last year production was increased to 65,500 cars. What was the increase expressed as a per cent?

25. The thickness of a sheet of steel as measured by a micrometer is 0·182″. The specifications call for 0·175″. Express the error as a per cent of the specified measurement.

Percentage

In order to meet the expenses of his business together with wages for himself and his employees a dealer estimates that the profit on the goods handled by him should be 20 per cent of the cost price of the goods. At this rate, what would be the profit on goods which cost $140?

Since 20% is equivalent to $\frac{20}{100}$

then his profit in dollars is $\frac{20}{100}$ of $140 = 28$.

or

Since 20% is equivalent to 0·20
then his profit in dollars is $0{\cdot}20 \times 140 = 28$.

Example 1: You are purchasing an overcoat the marked price of which is $45. The merchant promises to reduce the marked price by 15%. How much is the reduction?

Solution: The reduction in dollars is $\frac{15}{100} \times 45$

$= 6{\cdot}75$.

Example 2: A 300-lb. brass casting is 65% copper and 35% zinc. What weight of each metal does it contain?

Solution: The weight of copper in pounds is $\frac{65}{100} \times 300$

$= 195$.

The weight of zinc in pounds is $\frac{35}{100} \times 300$

$= 105$.

Check: $195 + 105 = 300$.

Exercise

A

1. Find the value of the following:

(*a*) 10% of 65, (*b*) 3% of 500, (*c*) 2% of 240,
(*d*) 6% of 8,000, (*e*) 5% of 3,000, (*f*) 1% of 35,
(*g*) 20% of 1,600, (*h*) 99% of 300, (*i*) 10% of 1.

2. Change the following per cents to equivalent decimals:

(*a*) 6%, (*b*) 15%, (*c*) 40%,
(*d*) 52%, (*e*) 75%, (*f*) 90%,
(*g*) 11%, (*h*) $12\frac{1}{2}$%, (*i*) $\frac{5}{8}$%,
(*j*) $62\frac{1}{2}$%, (*k*) $1\frac{1}{4}$%, (*l*) $\frac{1}{4}$%,
(*m*) $\frac{1}{2}$%, (*n*) 0·1%, (*o*) 0·3%,
(*p*) $\frac{1}{8}$%, (*q*) 125%, (*r*) 300%.

3. Change the following per cents to equivalent common fractions:

(*a*) 10%, (*b*) 30%, (*c*) 25%,
(*d*) 65%, (*e*) $14\frac{2}{7}$%, (*f*) 6%,

(*g*) 1%, (*h*) $12\frac{1}{2}$%, (*i*) $16\frac{2}{3}$%,
(*j*) $66\frac{2}{3}$%, (*k*) $11\frac{1}{9}$%, (*l*) $33\frac{1}{3}$%,
(*m*) 0·2%, (*n*) 200%, (*o*) 2·25%.

4. (*a*) Select from those per cents given in Question 3 at least four examples of per cents which cannot be readily expressed as decimal fractions.
(*b*) Suggest five other such per cents.

B

5. A worker receives $12 per day. If he receives a cut of 10%, find the reduction in his daily wage.

6. A clerk in a store receives $5·60 per day. Her wages are increased by 10%. What is her new wage?

7. A skilled mechanic whose wages a year ago were $15 per day received a 10% increase. What were his increased wages? Last week he received a 10% decrease. What will he now receive? Explain why there is a difference between the final wage and the original wage.

8. If bearing metal is 85% copper, 10% tin and 5% lead, how many pounds of each should be purchased to make 250 lb. of this metal?

9. An arithmetic paper was written by 250 students of whom $\frac{2}{5}$ were girls and the remainder boys. If 10% of the boys failed and 30% of the girls failed, how many students were successful?

10. A bronze bearing weighing 500 lb. is 8% tin, 15% lead and 77% copper. What weight of each of these metals must be purchased to make the bearing?

11. During the winter, 95% of the trees in a 25-acre reforested woodlot in Huron County, Ontario, were girdled and killed by mice. Calculate to the nearest hundred trees the number lost if there were 685 growing trees per acre the previous autumn.

12. During a water crisis in a large American city a survey showed that 9·4% of the water pumped was wasted because of leaky taps. If you assume the same rate of loss, how much of the 28,059,210,000 gallons of water pumped by an Ontario city was wasted from this cause?

Take-Home Pay

A machinist accepted a position at $60 per week. Like other wage earners he is required to make payments for unemployment insurance and income tax. What is his actual weekly income after providing for:

(*a*) unemployment insurance at the rate of 1% of the wage,
(*b*) income tax according to the following scale of rates:

On the part of the annual income up to $1,000 there is no tax.
On the part of the income from $1,000 to $2,000 the tax rate is 15%.
On the part of the income $2,000 to $4,000 the tax rate is 17%.

Solution: His yearly wage in dollars is ..... $52 \times 60 = 3120$
Unemployment insurance in dollars is ..... $0{\cdot}01 \times 3120 = 31{\cdot}20$
Tax on the first \$1,000 is nil
Tax on the second \$1,000 is $0{\cdot}15 \times 1000 = 150{\cdot}00$
Tax on remaining \$1,120 is $0{\cdot}17 \times 1120 = 190{\cdot}40$
Total income tax in dollars is $340{\cdot}40$
Total deduction for the year in dollars is $31{\cdot}20 + 340{\cdot}40 = 371{\cdot}60$
Balance of the yearly income in dollars is $3120 - 371{\cdot}60 = 2748{\cdot}40$

$$\text{Net weekly income in dollars is } \frac{2748{\cdot}40}{52}$$

$$= 52{\cdot}853.$$

Take-Home weekly wage is \$52·85.

Cost of Shelter

A man considers purchasing a home for \$10,000. Before making the purchase he endeavours to estimate the monthly cost of owning this home. After some investigation he obtains the following information:

(*a*) The money required for the purchase could be invested otherwise at 5% per annum.
(*b*) The property in this district is assessed at 60% of its sale value and the tax rate for the present year is 41 mills on the dollar.
(*c*) The house should be insured for \$8,400 against loss by fire at the rate of 70 cents per \$100 for 3 years.
(*d*) He estimates that repairs, redecorating etc. will cost him \$200 annually.

Using this information he could estimate the monthly cost of shelter as follows:

Interest on investment in dollars for 1 year is $0{\cdot}05 \times 10000 = 500{\cdot}00$
Assessment on property in dollars is $0{\cdot}60 \times 10000 = 6000{\cdot}00$
Annual taxes at 41 mills on dollar is $0{\cdot}041 \times 6000$ ..... $= 246{\cdot}00$
Insurance premium for 1 year is $\frac{1}{3} \times 0{\cdot}70 \times \frac{8400}{100}$ ..... $= 19{\cdot}60$
Repairs etc. annually in dollars ..... $= 200{\cdot}00$
Estimated annual cost in dollars is ..... $965{\cdot}00$

$$\text{Cost per month in dollars is } \frac{965{\cdot}60}{12}$$

$$= 80{\cdot}47.$$

Exercise

B

1. A man receives a salary of \$3,000 per year, payable in 12 equal monthly payments. Deductions are made by his employer as follows:

(i) 5% of wages for "retiring fund";
(ii) $1·50 per month for hospitalization plan;
(iii) $20 per month for income tax.

What is the net amount of his monthly pay?

2. Calculate the annual income tax payable by each of the individuals whose income and exemptions are listed below. Use the following table of rates:

on the first $1,000 or part thereof, of the taxable income, the rate is 15%;
on the next $1,000 or part thereof 17%;
on the next $2,000 or part thereof 19%;
on the next $2,000 or part thereof 22%;
on the next $2,000 or part thereof ..... 26%.

Individual	*Annual income in dollars*	*Total exemption in dollars*
A	2,500	1,350
B	1,800	1,240
C	4,500	2,700
D	4,500	1,650
E	10,000	3,170

3. A householder's property is assessed for $5,700. If the municipal tax rate is $35\frac{1}{2}$ mills on the dollar, what is the amount of tax payable? If $1\frac{1}{2}$% is deducted for prompt payment, what is the net tax?

4. A town having a total assessment of five million dollars built an addition to the school. It is to be paid for in ten annual equal instalments of $12,500.
   (*a*) By how much will this outlay increase the annual tax rate?
   (*b*) By how much will this expenditure increase the annual taxes on a home assessed for $1,800?

5. What will be the premium paid for insuring a home for $8,000 if the rate is 65 cents per $100 for 3 years?

6. Property valued at $10,000 is insured for 90% of its value at a rate of 70 cents per $100 for 3 years. What would be the annual cost for insurance?

Per Cent in Retail Business

Within the next six months a certain retail merchant expects to be able to sell 10 four-burner electric stoves. Let us consider the procedure he might follow to have these available for his customers at a fair price.

Making the Purchase

He consults the manufacturer's catalogue and finds the *list price* to be \$200 less a trade discount of 40%. He knows that to make this purchase he will have to borrow \$1,200 from the bank for a period of 6 months at an interest rate of 8% per annum and that other expenses—delivery charges, etc.—involved in this purchase can be estimated at \$50.

This information enables him to calculate the *prime cost* and *gross cost* of each stove as follows:

List price per stove in dollars is 200·00
Discount per stove in dollars is $0{\cdot}40 \times 200 = 80{\cdot}00$
Prime cost per stove in dollars is $200 - 80 \ldots\ldots = 120{\cdot}00$
Sales tax at 8% of invoice is $0{\cdot}08 \times 120 \ldots\ldots = 9{\cdot}60$
The interest on loan per stove in dollars is $\frac{1}{10} \times \frac{0{\cdot}08}{2} \times 1200 = 4{\cdot}80$
For one stove the other expenses in dollars are $\frac{1}{10} \times 50 = 5{\cdot}00$
Gross cost per stove in dollars is $120 + 9{\cdot}60 + 4{\cdot}80 + 5{\cdot}00 = 139{\cdot}40$

Estimating the Selling Price

He considers a fair rate of profit on stoves of this type to be 20% of the gross cost price. Thus he can calculate the *net selling price.*

$$\text{The net selling price in dollars is } \frac{120}{100} \times 139{\cdot}40$$
$$= 167{\cdot}28.$$

There are, however, considerable *overhead expenses* connected with the selling of this merchandise. These include rent, light, heat, advertising, employees' wages, etc. He has found from experience that these will be about 25% of the calculated net selling price of his merchandise. This enables him to calculate the share of overhead expenses to be included in the *gross selling price* of each stove.

The overhead expenses per stove in dollars are $0{\cdot}25 \times 167{\cdot}28 = 41{\cdot}82$.
The gross selling price per stove in dollars is $167{\cdot}28 + 41{\cdot}28 = 209{\cdot}10$.

Special Sale

Six months later one of these stoves remained unsold. The merchant decided to offer a 20% discount on its marked price. (*a*) What is its reduced selling price? (*b*) How much did he gain or lose on this stove?

Discount in dollars is $0{\cdot}20 \times 209{\cdot}10 = 41{\cdot}82$.
Reduced selling price in dollars is $209{\cdot}10 - 41{\cdot}82 = 167{\cdot}28$.
Gross cost price of each stove in dollars is 139·40.
Gross selling price of this stove in dollars is 167·28.
Estimated selling expenses on each stove in dollars are 41·82.
Net selling price of this stove in dollars is $167{\cdot}28 - 41{\cdot}82 = 125{\cdot}46$.
Loss on this stove in dollars is $139{\cdot}40 - 125{\cdot}46 = 13{\cdot}94$.

Definitions: **List Price** of an article is the price stated in a catalogue which is sent to the dealer by a manufacturer or a wholesale firm.

Trade Discount is the amount by which the list price is reduced when goods are sold to a dealer. The rate of discount is expressed as a per cent of the list price.

Net Cost Price is the actual amount paid for the goods. Sometimes this is called the **prime** cost.

Gross Cost is the amount paid for the goods plus the expenses involved in the purchase. It includes such items as sales tax, freight charges, etc.

Gross Selling Price is the actual price at which the goods are sold.

Net Selling Price is the gross price less the expenses involved in selling.

Profit or Loss is the difference between the Gross Cost Price and the Net Selling Price.

Exercise B

1. The list price of an article is \$250. What is the net price if a discount of 30% is given?
2. In a manufacturer's catalogue an article is listed at \$400, with a discount of 25%. To encourage prompt payment an additional discount of 10% is given if the account is paid within 30 days. What will be the net cost price of this article if (*a*) payment is not made within 30 days; (*b*) payment is made within 30 days?
3. The net cost price of an article is calculated to be \$36. If there is a sales tax of 8%, and transportation and the other purchasing charges amount to \$6·40, what is the gross cost of this article?
4. The gross cost of an article is \$120. If a profit of 25% is expected what will be the net selling price?
5. The net selling price of an article is \$75. What will be the gross selling price if the overhead is estimated at 20% of the net selling price?
6. A merchant offers goods for sale on a time-payment plan which requires 25% of the purchase price to be paid at the time of purchase and the balance in 10 equal monthly instalments. Find the down payment and each of the monthly payments for goods which are priced at (*a*) \$80, (*b*) \$250, (*c*) \$144.
7. A customer who borrows \$100 from the bank for 1 year agrees to pay interest at the rate of 6% per annum. If the amount of interest is deducted from the loan how much does the customer actually receive?
8. From his records for the past year a merchant finds that his overhead amounted to \$24,000 and his total sales amounted to \$75,000. What per cent of his gross sales are required for overhead expenses?

9. The marked price of an article was \$25·60. If it was sold for \$24, what was the rate of discount?

Commission

A man considers accepting a position in the sales department of a certain firm. He may choose one of the following plans for determining his annual wage:

Plan A: He is to receive an amount equal to 5% of all his sales.

Plan B: He is to receive an annual salary of \$1,500 plus 2% of all his sales.

Which plan will give him the larger income if his sales amount to (*a*) \$45,000, (*b*) \$52,000?

Solution: (*a*) The sales are \$45,000. When plan A is used, the agent's income in dollars is $0{\cdot}05 \times 45000 = 2250$.

When plan B is used, the commission on sales in dollars is

$$0{\cdot}02 \times 45000 = 900.$$

His annual salary in dollars is $1500 + 900 = 2400$.

Therefore, if the sales are \$45,000 plan B is the better for the agent by \$150.

(*b*) The sales are \$52,000. When plan A is used, the agent's income in dollars is $0{\cdot}05 \times 52000 = 2600$. When plan B is used, the commission on sales in dollars is $0{\cdot}02 \times 52000 = 1040$.

His annual salary in dollars is $1500 + 1040 = 2540$.

Therefore, if the sales are \$52,000, plan A is better for the agent by \$60.

Can you find for what amount of sales each plan would give the same income?

Exercise

B

1. A boy sold 200 boxes of Christmas cards at 50 cents each. If his rate of commission was 20%, how much did he receive for his efforts? How much did he remit to his employer?

2. In addition to a regular wage a departmental store gave its salesmen a commission of $\frac{1}{2}$% on their total sales. How much extra did each of the following members of the sales staff receive for a week in which their sales were as follows:

(*a*) A's sales, \$1,000; (*b*) B's sales, \$800; (*c*) C's sales, \$875?

3. A real-estate agent sells a house for \$8,000. His rate of commission is $3\frac{1}{2}$%. What is his commission? How much did the owner actually receive for his house?

4. An insurance agent receives a commission of 3% of the face value of the policies he sells. What will be his commission on policies with the following face values:

(*a*) \$1,000, (*b*) \$25,000, (*c*) \$3,000?

5. A real-estate firm charges a commission of 5% on the selling price of a home. The actual agent that sells the house receives 70% of the charge made by the real-estate firm. How much would the agent receive if he sold a house for $13,000?
6. A boy sells magazines during the summer. In one week he sold 23 subscriptions for magazine A at $2 each; 14 subscriptions for magazine B at $·75 each; and 5 subscriptions for magazine C at $4·50 each. If he is allowed a 30% commission, how much did he make during that week?
7. To promote the sale of tickets for an exhibition the committee announce that they will grant a commission of 9% on the face value of all tickets sold over and above 10. If a boy sells 35 tickets at 50 cents each, what would be his commission?

Review Exercise

B

1. A man plans to invest 30% of his capital in bonds, 40% of it in industrial stocks and the remainder in mining stocks. If he has $16,000 how much will he invest in each type?
2. A man leaves an estate of $80,000 to be divided as follows: to named charities 10%; to his wife 60% of the balance; to his son 60% of the remainder and the balance to his daughter. How much did each receive?
3. Express the following per cents as fractions in their lowest terms:
   (*a*) 8%, (*b*) 35%, (*c*) $\frac{1}{2}$%, (*d*) 225%,
   (*e*) $6\frac{1}{2}$%, (*f*) 0·015%, (*g*) $1\frac{1}{4}$%, (*h*) 0·8%.
4. (*a*) What per cent is 6 of 8? (*b*) What per cent is 5 of 9?
   (*c*) What per cent is 15 of 10? (*d*) What per cent is 1·9 of 2·3?
   (*e*) What per cent is 0·015 of 0·315?
5. The total enrolment of a school is 1,610 pupils. The average attendance for a month was 94·6%. On the average, how many attended school each day?
6. A boy takes mathematics for 6 periods out of a possible 40 periods. What per cent of his time is spent in the mathematics class?
7. A store-keeper purchased 50 pairs of rugby shoes for sale in the fall of the year. He paid the wholesaler $8 per pair. The wholesaler added on the 8% sales tax.
   (*a*) What was the total cost to the retailer?
   (*b*) To pay for these shoes the retailer borrowed the money from the bank at 6% for 3 months. How much interest did he pay on his loan?
   (*c*) If he wishes to make 20% on his gross cost, find the price per pair at which they will have to be sold.
8. A salesman sells cars on a commission of 8%. During a week he sold cars for $2,125, $875, and $350. What was his commission for the week's sales?

9. A firm which deals with credit union members allows a discount of 15% on articles purchased from them. What discount would a member expect to get on

(*a*) an iron costing \$14·75; (*b*) a chesterfield costing \$275?

10. During a sale a store-keeper allowed a discount of 8% of the marked price of goods. If a baseball mitt was marked \$4·75, what would be its sale price?

11. A book store advertised "30% off all stock". A lady purchased 3 books marked \$1·98, \$·69 and \$2·49. What should she pay if she is allowed the discount?

12. A single man receives a weekly wage of \$60. He is allowed an exemption of \$1,000 per year. Calculate the tax he should pay according to the following schedule:

Taxable Income	*Tax*
\$1,000 or less	15%
\$1,000	\$150 plus 17% on the next \$1,000.
\$2,000	\$320 plus 19% on the next \$2,000.
\$4,000	\$700 plus 22% on the next \$2,000.

13. A man purchases a house for \$13,000 and pays \$5,000 cash. He gives a mortgage for the balance bearing interest at 5% per annum payable yearly. He agrees to reduce the principal by \$500 at the end of each year. Calculate each of his first five annual payments.

14. In one year in Ontario there were 1,035 forest fires reported by the fire-rangers. The next year the number increased to 1,800.

(*a*) By what per cent did the number of fires increase?

(*b*) It is estimated that 15% of forest fires are due to natural causes but all others are attributed to human carelessness and are avoidable. At this rate how many of the 1,800 fires were caused by human carelessness?

CHAPTER 6

Measurement of Distance

Standard Linear Units of Measurement

There are in use today two universally recognized systems of units of measurement: the Metric system and the English system. The basic unit of length in the Metric system is the **metre**; that in the English system is the **yard**, which is slightly shorter than a metre. A **yard** in Canada is defined by statute to be $\frac{9144}{10000}$ of an International Metre. The *standard metre* is preserved in the Archives of the International Metre Commission at Sèvres, near Paris, France. Replicas of all standard units are kept for reference by National Bureaus of Standards in most countries of the world.

The two systems differ radically in the methods employed to derive the other units from the basic unit. In the Metric system, each derived unit less than the metre is obtained by dividing this basic unit by some power of ten. Each derived unit greater than the basic unit is obtained by multiplying the basic unit by some power of ten.

The English system was not so simply developed. It is a strange mixture of custom and convenience which gradually evolved under the influence of traders, soldiers and diplomats from many lands. Even some of the names are suggestive of primitive units of measurement: the hand, the width of the human hand (about 4″); the span, the distance from the end of the thumb to the little finger when the hand is stretched out; or the ell, the length of cloth wound around the hand and elbow. Tradition would have us believe that the yard was the distance from a man's nose—that of Henry I of England —to the thumb of his outstretched arm. Many such units have outlived their usefulness and are now obsolete. Such units as the cubit—about 18 inches; the Roman pace—about 5 feet; and the ell have become merely names associated with the past. The modern conception of the pace as equivalent to one step is approximately $2\frac{1}{2}$ feet.[1]

In the English Linear System some units are common to all walks of life, some are more specialized. By knowing well the basic relationships—defined by law—between the units, other relationships can be calculated.

[1] Bendick, Jeanne, *How Much and How Many*, McGraw-Hill Book Co. Inc., p. 163 ff.

ENGLISH LINEAR UNITS OF MEASUREMENT

12 inches (in.) = 1 foot (ft.).
3 feet = 1 yard (yd.).
$5\frac{1}{2}$ yards = 1 rod (rd.).
320 rods = 1 statute mile (mi.).

Specialized Units

Navigators measure distances at sea in **nautical miles**.

1 fathom (for depth of water) = 6 feet.
1 nautical mile (n. mi.) = 6,080 feet.
1 knot is a sailing rate of 1 nautical mile per hour.
1 nautical mile is the distance, along a meridian, a ship travels for each change of 1 minute of latitude.

Land surveyors use the **link** and the **chain** for determining the boundaries of properties.

1 chain = 66 feet.
1 chain = 100 links.

The chain as a unit of measurement, was introduced into England by Edward Gunter early in the seventeenth century. It is usually called Gunter's chain to distinguish it from the engineer's steel tape which is 100 feet long, each foot of which may be divided into tenths.

Electricians use the **mil** to express the measure of the diameter of a wire.

1 mil = $\frac{1}{1000}$ inch.

Printers use the **point system** of units of measurement for expressing the space requirements for printed matter.

A point is a small unit of length about the diameter of a dot (.) that is made with a sharp pencil. Actually,

1 point = 0·013833 inch (very nearly $\frac{1}{72}''$).
1 inch is equivalent to 72 points.
6 points = 1 nonpareil.
12 points = 1 pica.

Exercise

A

1. Express the distance of 1 mile in each of the following units:
(*a*) yards, (*b*) feet, (*c*) inches, (*d*) chains.

2. Express the distance of a rod in each of the following units:
(*a*) mile, (*b*) foot, (*c*) chain, (*d*) link.

3. Express the distance of 1 yard in each of the following units:
(*a*) mile, (*b*) rod, (*c*) inch, (*d*) chain.

4. Express the distance of 1 chain in each of the following units:
(*a*) rod, (*b*) yard, (*c*) inch.

5. Complete the following table:

(*a*) 24 inches	= ... feet.	(*b*) 24 inches	= ... yards.
(*c*) 18 inches	= ... feet.	(*d*) 18 inches	= ... yards.
(*e*) 11 yards	= ... rods.	(*f*) 440 yards	= ... miles.
(*g*) 5 chains	= ... links.	(*h*) 660 feet	= ... chains.
(*i*) 1 n. mile	= ... feet.	(*j*) 5 mils	= ... inches.
(*k*) 18 mils	= ... inches.	(*l*) 0·008 inch	= ... mils.
(*m*) 1·006 inches	= ... mils.	(*n*) 1 rod	= ... feet.

B

6. The maximum speed allowed on Ontario highways is 60 miles per hour. At this speed, how many feet will a car travel in 1 second?

7. A boat sails from Halifax at an average speed of 28·5 knots. In $8\frac{1}{2}$ hours, how many nautical miles will it be from Halifax?

8. Without referring to the tables of linear units of measurement, copy and fill in the blanks in each of the following statements:

(*a*) 3 ft.	= ... inches.	(*b*) 9 inches	= ... ft.
(*c*) 3·5 in.	= ... mils.	(*d*) 253 mils	= ... in.
(*e*) $2\frac{1}{7}$ ft.	= ... yd.	(*f*) $\frac{5}{9}$ yd.	= ... ft.
(*g*) $\frac{1}{2}$ rd.	= ... yd.	(*h*) $3\frac{2}{3}$ yd.	= ... rd.
(*i*) 2·5 mi.	= ... rd.	(*j*) 400 rd.	= ... mi.
(*k*) 5 mi.	= ... yd.	(*l*) 2640 ft.	= ... mi.

9. Without referring to the tables, copy and fill in the following statements:

(*a*) 1·25 mi.	= ... ch.	(*b*) 200 ch.	= ... mi.
(*c*) 10 links	= ... in.	(*d*) 20 fathoms	= ... ft.
(*e*) 10 n. mi.	= ... mi.	(*f*) 100 mi.	= ... n. mi.
(*g*) 132 n. mi.	= ... mi.	(*h*) 760 mi.	= ... n. mi.

10. Express:

(*a*) 25 ft. 3 in. as feet; (*b*) 25 yards in rods;
(*c*) 32 feet in fathoms; (*d*) 35 yd. 2 ft. 8 in. as yards;
(*e*) 3·2 feet in feet and inches.

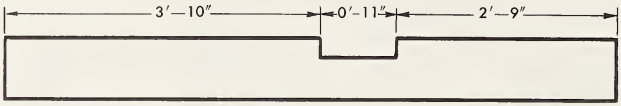
11. Express each of the compound measurements as a simple one, using the unit of measurement indicated:

(*a*) 10 ft. 3 in. in inches, (*b*) 5 yd. 2 ft. in yards,
(*c*) 10 mi. 75 rd. in rods, (*d*) 7 ft. 8 in. in feet.

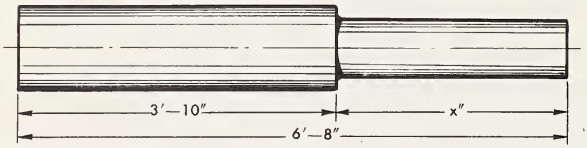
12. Express each of the following simple measurements as a compound measurement in the units indicated:

(*a*) 245 inches in feet and inches;
(*b*) 253 inches in yards, feet and inches;
(*c*) 500 rods in miles and rods.

13. Find the over-all length, expressed in feet and inches, represented in the drawing.



14. Find the value of x in the diagram below.



15. Find the sum of the distances:

(*a*) 3 ft. 7 in., 9 ft. 8 in., 7 ft. 11 in.
(*b*) 14 ft. $3\frac{1}{4}$ in., 8 ft. $9\frac{1}{2}$ in., 7 ft. $10\frac{1}{8}$ in.
(*c*) 36′ $5\frac{7}{32}$″, 52′ $8\frac{11}{16}$″, 38′ $7\frac{7}{8}$″.

16. Subtract 3 ft. 10 in. from 8 ft. 9 in.

17. How much greater is 15′ 8″ than 5′ 11″?

18. What must be added to 5′ $9\frac{1}{2}$″ to make it equal to 8′ $7\frac{1}{4}$″?

19. It is estimated that a car driver usually requires $\frac{3}{5}$ second from the time he sees a road hazard until he can take any action toward stopping the car. How many feet will the car have travelled during this time at each of the following speeds:

(*a*) 60 miles per hour; (*b*) 45 miles per hour;
(*c*) 40 miles per hour; (*d*) 30 miles per hour;
(*e*) 20 miles per hour.

Why does a speed limit of 30 miles per hour seem reasonable for urban areas?

C

20. What is the measure of the distance of 1 nautical mile when it is expressed in statute miles? Express this measure

(*a*) using a common fraction in its lowest terms;
(*b*) using a decimal fraction correct to three places of decimals.

21. A boat is travelling in a canal at a speed of 20 knots. What would be the speed in statute miles per hour of an automobile which, travelling along a road parallel to the canal, just kept up with the boat?

22. Using the nautical mile as the unit of measurement, express the measure of the distance of 1 statute mile (*a*) in fractional form, (*b*) in the form of a decimal fraction to the nearest thousandth.

23. A ship's navigator is informed that the velocity of the wind is 30 statute miles per hour. Express this rate in nautical miles per hour.

24. Make a neat sketch of a 2-inch length of a scale graduated in inches, picas and nonpareils.

25. A page of printing 6 inches in depth is to contain 54 lines. What size of type must be used? The size of the type is the measure, in points, of the depth required for 1 line.

26. How many lines of 8-point type can be set in a column 11 inches long?

27. According to a survey map, a rectangular township is 1,680 chains long and 1,240 chains wide. Find its length and width in miles.

28. Express:

(*a*) 30 chains 24 links as chains; (*b*) 45 rods as chains and links;
(*c*) 17·3 inches as links to the nearest hundredth;
(*d*) 6 miles 40 rods as chains; (*e*) 5 links as chains.

The Metric System

The Metric system of units of measurement originated in France late in the eighteenth century. Since then it has been adopted as the commonly used system of measurement by practically all countries of the world except those of the British Commonwealth and the United States. Although it was officially adopted by the government of the United States in 1840, it was only in the system of units of money that it came into common use. The Metric system is almost universally adopted in scientific work. Many leaders in industry, commerce and government as well as in science are desirous of having the Metric system of measurement adopted for all purposes.[2]

The simplicity of the Metric system of units of measurement depends upon certain characteristics, three of which are mentioned here:

(1) For each quantity to be measured there is a basic unit; for distance, the metre; for weight, the gram; for capacity, the litre; for electric current, the ampere; etc.

(2) Other units are derived from the basic unit by multiplying or dividing by some power of 10.

(3) The name of each derived unit is formed by adding to the basic name a prefix which indicates the relationship to the basic unit. The same set of prefixes are used in most tables in the Metric system.

---

[2] *Twentieth Year Book*—National Council Teachers of Mathematics—The Metric System of Weights and Measures.

METRIC UNITS OF LINEAR MEASUREMENT

Prefix	*Meaning of prefix*	*Relationship to the basic unit*
kilo	1000	1 kilometre (Km.) = 1000 metres.
hecto	100	1 hectometre (Hm.) = 100 metres.
deka	10	1 dekametre (Dm.) = 10 metres.
The basic unit for linear measure is 1 metre (m.)		
deci	$\frac{1}{10}$	1 decimetre (dm.) = 0·1 metre.
centi	$\frac{1}{100}$	1 centimetre (cm.) = 0·01 metre.
milli	$\frac{1}{1000}$	1 millimetre (mm.) = 0.001 metre.

Of the units larger than a metre, only the kilometre is in common use. It is the metric unit corresponding to the English mile.

The Metric Linear Relationships may also be stated as follows:

10 millimetres = 1 centimetre.
10 centimetres = 1 decimetre.
10 decimetres = 1 metre.
10 metres = 1 dekametre.
10 dekametres = 1 hectometre.
10 hectometres = 1 kilometre.

Exercise

A

1. Study the metric scale on your ruler. Locate the line marking the end of 1 centimetre. Into how many equal parts is each centimetre divided? How many millimetres in 1 centimetre? How would you express a distance of 6 cm. 3 mm. as centimetres; as millimetres? Where is this measurement indicated on your ruler? How does 1 mm. compare in size to $\frac{1}{16}''$ on the English scale?

2. Express:
(*a*) 3 m. in cm., 0·8 m. in cm., 2·75 m. in cm.;
(*b*) 30 mm. in cm., 2·4 mm. in cm., 876 mm. in cm.;
(*c*) 300 cm. in m., 567 cm. in m., 34 cm. in m.;
(*d*) 5 Km. in m., 75 Km. in m., 0·198 Km. in m.;
(*e*) 5 cm. in mm., 34·78 cm. in mm., 0·009 cm. in mm.;
(*f*) 2 m. in mm., 0·34 m. in mm., 0·087 m. in mm.;
(*g*) 40 m. in Km., 3,456 m. in Km., 0·9 m. in Km.;
(*h*) 34 cm. in dm., 267·8 cm. in dm., 0·09 cm. in dm.;
(*i*) 350 m. in Dm., 467 Dm. in m., 45 Dm. in dm.;
(*j*) 2·4 Hm. in Km., 2·56 Hm. in m., 0·008 Hm. in mm.

3. If 1 inch = 2·54 centimetres, express the following lengths in centimetres:
(*a*) 2″, (*b*) 10″, (*c*) 100″, (*d*) 8′ 4″,
(*e*) 0·5″, (*f*) 1′ 8″, (*g*) $\frac{1}{4}$″, (*h*) $\frac{1}{10}$″.

4. If 1 in. = 2·54 cm., how would you find the number of inches in 1 centimetre?

5. If 1 in. = 2·54 cm., how would you find the number of centimetres in 1 yard? The number of inches in 1 metre?

6. One kilometre is approximately $\frac{5}{8}$ mile. Find the distance in miles of
(*a*) 16 Km., (*b*) 80 Km., (*c*) 640 Km., (*d*) 24 Km.

7. Find the number of kilometres in
(*a*) 50 miles, (*b*) 60 miles, (*c*) 40 miles, (*d*) 30 miles.

B

8. Express 1 centimetre in inches to 4 significant digits.

9. Express one metre in inches to 4 significant digits.

10. Express in centimetres:
(*a*) 25 inches, (*b*) 36 inches, (*c*) 1·25 inches.

11. Express in inches:
(*a*) 65 centimetres, (*b*) 28·6 centimetres, (*c*) 0·89 centimetres.

12. Express in kilometres:
(*a*) 35 miles, (*b*) 7·5 miles, (*c*) 340 miles.

13. Express in miles:
(*a*) 24 kilometres, (*b*) 18·6 kilometres, (*c*) 3,240 kilometres.

14. Without reference to the table of metric measurement, copy and fill in each of the following statements:

(*a*) 5 m.	= ... dm.	(*b*) 3·2 dm.	= ... cm.
(*c*) 5·1 cm.	= ... dm.	(*d*) 75 mm.	= ... cm.
(*e*) 5 cm.	= ... dm.	(*f*) 234 dm.	= ... m.
(*g*) 25 m.	= ... cm.	(*h*) 2·53 m.	= ... mm.
(*i*) 3·7 dm.	= ... mm.	(*j*) 273 mm.	= ... m.
(*k*) 2·3 Km.	= ... m.	(*l*) 53·4 m.	= ... Km.
(*m*) 0·035 Km.	= ... m.	(*n*) 0·058 Km.	= ... cm.
(*o*) 356·5 cm.	= ... Km.	(*p*) 0·0875 m.	= ... mm.
(*q*) 0·575 cm.	= ... m.	(*r*) 8·35 mm.	= ... dm.
(*s*) 3,000 mm.	= ... Km.	(*t*) 26·5 dm.	= ... Km.

15. (*a*) Express a distance of 89 feet in inches.
(*b*) Express a distance of 98 dm. in cm.
(*c*) Express a distance of 5,376 yd. in miles.
(*d*) Express a distance of 5,376 m. in kilometres.

16. Express each of the following compound measures as a simple measurement in the unit indicated:

(*a*) 35 m. 5 dm. in (i) metres, (ii) decimetres;
(*b*) 5 m. 3 cm. in (i) metres, (ii) centimetres;
(*c*) 3 dm. 5 cm. in (i) metres, (ii) centimetres, (iii) millimetres;
(*d*) 5 Km. 2 m. in (i) metres, (ii) kilometres;
(*e*) 7 Km. 3 m. 5 cm. in (i) metres, (ii) centimetres, (iii) kilometres.

17. Considering the relationship 1 inch = 2·54 cm. to be exact, find in decimal notation:

(*a*) the measure of 1 kilometre in miles correct to 3 significant digits;
(*b*) the measure of 1 mile in kilometres correct to 3 significant digits.

18. Express each of the following measurements as compound measurements using the units indicated:

(*a*) 25 millimetres in centimetres and millimetres;
(*b*) 453 mm. in decimetres, centimetres and millimetres;
(*c*) 7563 m. in kilometres and metres;
(*d*) 245·3 dm. in metres, decimetres, and centimetres.

19. Express in metres and find the total length of 3 lines which measure 3,457 mm., 20·6 dm. and 4,510 cm.

20. Add 35·27 m., 711 cm. and 890 mm.

21. Copy and fill in the blanks in each of the following statements:

(*a*) 3 in.	= ... cm.	(*b*) 1 ft.	= ... cm.
(*c*) 1 yd.	= ... cm.	(*d*) 5 cm.	= ... in.
(*e*) 1 dm.	= ... in.	(*f*) 1 m.	= ... in.
(*g*) 40 in.	= ... cm.	(*h*) 10 mi.	= ... Km.
(*i*) 20 Km.	= ... mi.	(*j*) 7·5 Km.	= ... mi.

22. Express the speed of 27 Km. per hour as centimetres per second.

23. A motorist on the highway travelled 100 yards in 4 seconds. What was his average speed in miles per hour?

24. A motorist in Brazil travelled a distance of 100 metres in 5 seconds. What was his average speed in kilometres per hour?

25. A gun that shoots a 75 mm. shell has a bore 75 mm. in diameter. What is the diameter of the bore in inches?

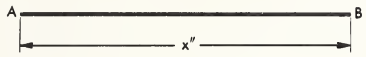
26. During the Second World War after the collapse of France a shipment of motor cycles intended for the French Army was sent instead to the Canadians in Great Britain. The speedometers were marked in kilometres per hour. If you had been riding one of these machines, how many miles per hour would you have been travelling if the speedometer read (*a*) 48 Km. per hour, (*b*) 100 Km. per hour?

27. Some boys can run 100 yd. in 10 seconds. If you are one of them, how long would it take you to run 100 metres at the same rate?

28. A military map was drawn to a scale of 1 inch representing 20,000 inches. When the map was read, a metric scale was the only one available and two places were found to be 5·7 cm. apart. How far were they apart in kilometres?

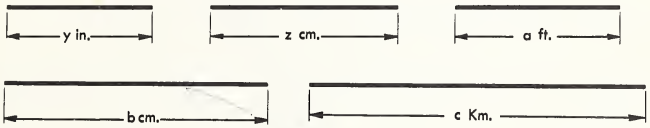
Measure Expressed by General Number

Draw a straight line segment of undetermined length and name it **AB**. Since we do not know the measure of its length, we may follow a common practice in mathematics and represent its measure in inches by x.

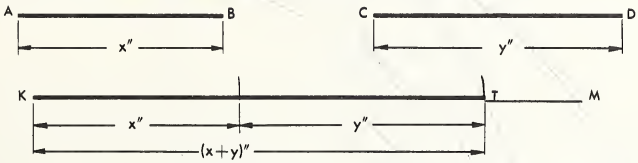


Notice that x, when used in this sense, is not a letter of the alphabet; it is a number.

Other lines of different lengths would have to have other numbers as the measures of their length. Their measures might be represented by the numbers y, z, a, b, c, t etc. as the occasion demands.

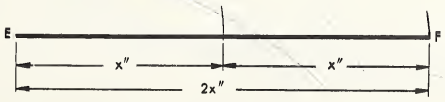


Suppose we draw two lines **AB** and **CD** with lengths in inches x and y respectively. Now from a new line **KM** we can step off with the compasses a line **KT** which is equal in length to the sum of **AB** and **CD**.



KT has a measure in inches of $x + y$.

We can make also a line **EF** which is twice as long as **AB**. Its measure in inches could be represented as $x + x$.



To find a simpler form of the measure $x + x$ consider the following expressions:

$$8 + 8 = 2 \times 8$$
$$17 + 17 = 2 \times 17$$
$$\tfrac{1}{2} + \tfrac{1}{2} = 2 \times \tfrac{1}{2}$$

Therefore, since x is a number

$$x + x = 2 \times x.$$

Now, the measure of **EF** in inches is $x + x$ or $2x$.

When arithmetical numbers are expressed as products it is necessary to put the multiplication sign between the factors 2×8 or the number would read 28 (twenty-eight). With numbers such as $2 \times x$ it is not necessary to write the multiplication sign. At all times it is understood that a number such as $2x$ means $2 \times x$. Similarly for other general measures which are written side by side the operation of multiplication must be understood.

Hence, $3y$ means $3 \times y$
$3ab$ means $3 \times a \times b$.

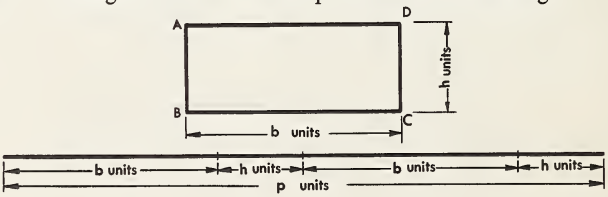
Exercise

A

1. If the length of a line is x feet, what is its length in inches?
2. If the length of a line is x inches, what is its length in feet?
3. The length of a line is y yards. What is its length in feet? in inches?
4. The length of a line **AB** is n centimetres. What is its length in millimetres? in metres?
5. If a line is z metres long, what is its length in centimetres?
6. If the cost of an article is c dollars, what is its cost in cents?
7. If a rectangle is x units long and y units wide, what is its area; its perimeter?

Perimeters of Rectangles and Squares

A perimeter is the measure of the distance round the outside of a plane figure. Since the perimeter is a measure of distance, it is expressed in linear units. Let us look at a general rectangle. Suppose the sides are formed of wire so that we can cut it at **B** and open it out into one long piece as in the diagram. The length of the wire is the perimeter of the rectangle.



The number of linear units in the perimeter is expressed by

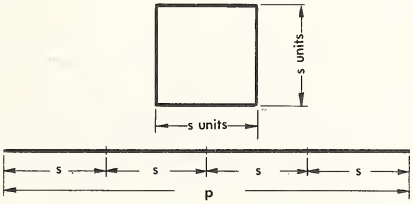
the sum	$b + h + b + h$
which can be simplified to	$2b + 2h$
or to	$2(b + h)$

Usually the measure of the perimeter in linear units is represented by the number p.

Now, using the same unit of measurement, we have the measure of the perimeter expressed in two different ways. These two expressions must be equal. Therefore, $p = 2(b + h)$.

This formula expresses the measure of the perimeter of any rectangle in terms of the measures of its length and width.

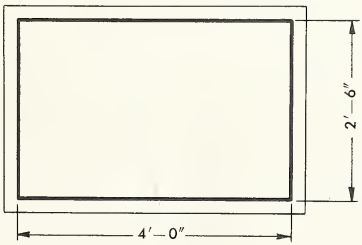
Now consider the square. If the number of linear units in each side is s and the number of linear units in the perimeter is p, what relationship exists between p and s?



The relationship is , $p = s + s + s + s$. This formula is usually written

$$p = 4s.$$

Example 1: A window screen is to be made 4 feet high and 2 feet 6 inches wide. How much half round will be needed to tack down the screen? (Use outside measurements.)

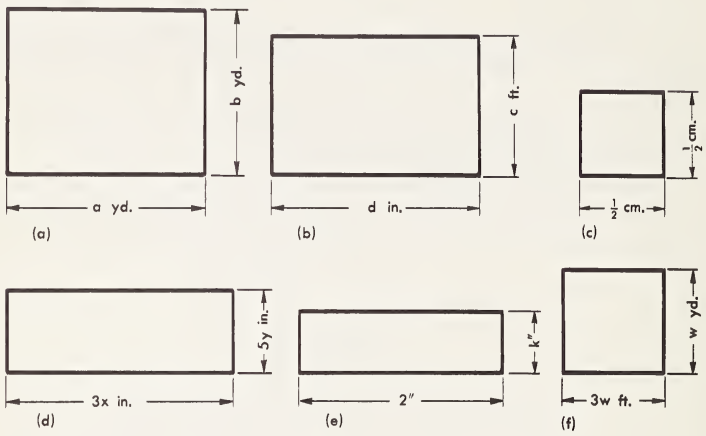


Solution: Height of the screen in feet is 4.
Width of the screen in feet is $2\frac{1}{2}$.
Perimeter of the screen in feet is $2(4 + 2\frac{1}{2})$
$= 2(6\frac{1}{2})$
Length of half round in feet $= 13$.

Exercise

A

1. In the following rectangles find the perimeter in a suitable unit of measurement.



B

2. Find the perimeter in feet of a rectangle 8′ 6″ long and 4′ 3″ wide.
3. Find the distance around a country block which is a square $1\frac{3}{4}$ miles to the side.
4. A square has a perimeter of p inches and each side is marked $\frac{1}{2}x$″. What is the relationship between the measures p and x?
5. The cross-section of a ventilating duct is a rectangle 14″ wide and 8″ high. What length of sheet metal will be required so that it will fold to form a length of this duct? Allow $\frac{1}{2}$″ for a lap seam.
6. Find the perimeter of a triangle if the sides are $6\frac{3}{4}$″, $5\frac{5}{16}$″ and $7\frac{1}{8}$″.
7. Find the perimeter of an isosceles triangle if each of the equal sides is 34·7 cm. and the base is 18·9 cm.
8. Find the perimeter of an equilateral triangle each side of which is 88·9 cm.
9. Find the semi-perimeter of a triangle with sides of 240·8 inches, 389·6 inches and 257·4 inches.
10. In a quadrilateral **ABCD**, **AB** = 17·6 cm., **BC** = 9·8 cm., **CD** = 14·6 cm. and **DA** = 19·9 cm. Determine the perimeter.
11. Find the perimeter of a quadrilateral with two sides each 32·5 inches long and the other sides 3·58 inches and 2·39 inches respectively.

Circumference of a Circle

Finding the relationship between the circumference and diameter of a circle is an ancient problem. When measurements were not very accurate, it was thought that the circumference was three times the diameter. In the Holy Bible, I Kings, vii: 23, we read:

> "And he made a molten sea, ten cubits from one brim to the other: it was round all about, . . . and a line of thirty cubits did compass it round about."

When more modern measuring instruments were introduced, a more accurate relationship than this was required.

By carefully carrying out the following experiment you can find for yourself the relationship between circumference and diameter of a circle to a greater degree of accuracy than that suggested in the Biblical reference.

Experiment: Obtain three or more articles that are circular in shape, such as a tin can, a cooking utensil, a mirror, etc. For each of these objects obtain the measure of its diameter and the measure of its circumference. To find the measure of the circumference, wrap a strip of paper around the object, mark the distance, open the paper and measure the length.

Let the measure of the circumference be represented by c.

Let the measure of the diameter be represented by d.

Record your results in a table similar to the one below. In each case calculate the quotient $\frac{c}{d}$.

Circumference in inches c	*Diameter in inches* d	$\frac{c}{d}$

You will note that the quotients are practically the same regardless of the size of the circle from which the measurements were taken. Actually, these quotients are the same for all circles. Mathematicians can calculate this number to any desired degree of accuracy. Since it is impossible to express this number exactly in our ordinary number system it is, for theoretical work, represented by the Greek letter π. Then the exact relationship between the diameter and the circumference can be expressed by the formula,

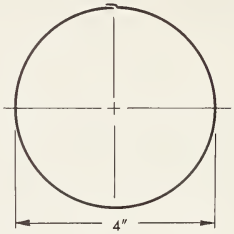
$$c = \pi d.$$

In this relationship c is the measure of the circumference in linear units and d is the measure of the diameter in the same linear units.

The following numbers are used for π, depending upon the accuracy required:

3 (rarely used),
$3\frac{1}{7}$ or 3·14 (commonly used),
3·1416 (where a higher degree of accuracy is justified).

Example: A circular can is to be constructed with a base 4″ in diameter. Find the length of the piece of tin needed to construct the sides if $\frac{1}{4}$″ is allowed for a seam.



Solution: Diameter in inches is 4.
Circumference of the can in inches is $3{\cdot}14 \times 4$
$= 12{\cdot}56$
Allowance for seam in inches is $0{\cdot}25$
Length of metal required in inches is $12{\cdot}81$
or 12·8 approx.

Exercise

A

1. Using $\pi = 3\frac{1}{7}$, state the circumference of the circle if the diameter is as follows:
   (*a*) 7″, (*b*) 14″, (*c*) 21″, (*d*) 28″, (*e*) 1·4″.
2. Determine the radius of a circle which has a diameter as follows:
   (*a*) $3\frac{1}{2}$″, (*b*) $7\frac{1}{2}$″, (*c*) $3\frac{3}{4}$″, (*d*) 1·19″.
3. Determine the diameter of a circle which has a radius as follows:
   (*a*) $1\frac{5}{16}$″, (*b*) 2·35″, (*c*) 0·875″, (*d*) $1\frac{1}{16}$″.
4. Using $\pi = 3{\cdot}14$ determine the circumference of a circle which has a diameter as follows:
   (*a*) 10″, (*b*) 20″, (*c*) 3″, (*d*) 12″, (*e*) 1·5″.
5. Determine the diameter of a circle which has a circumference as follows:
   (*a*) 3·14″, (*b*) 31·4″, (*c*) 6·28″, (*d*) 628 cm., (*e*) 94·2″.

B

6. Find the length of the steel rim of a wagon wheel which is 41 inches in diameter.
7. A circle has a radius of 3·5″. Calculate its circumference.

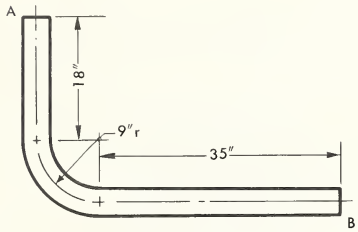
8. What length of label is required to go round a tin of peas the diameter of which is 3·1 inches? Allow $\frac{1}{4}$ inch for an overlap.

9. In each turn, what length of belt passes over an 8-inch pulley if there is no slipping?

10. What is the length of the tread on an automobile tire if the outside diameter is

(*a*) 28·4″, (*b*) 29″, (*c*) 30″.

11. What is the length of the conduit pipe from **A** to **B** in the diagram?



12. If Wilcox Lake is circular and has an average diameter of $\frac{1}{2}$ mile, how many feet of shore line would be on the lake? Estimate the number of lots with 100′ frontage on the lake.

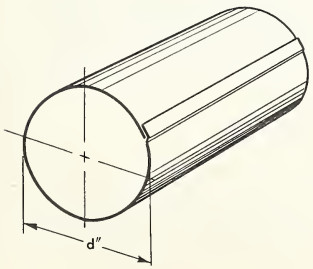
13. Some workmen needed to know the diameter of a water main, the ends of which were inaccessible. They found the circumference to be 20 inches. What was the diameter?

14. Find to the nearest tenth the measure of the diameter of each of the circles if the circumference is:

(*a*) 10″, (*b*) 50″, (*c*) 25 m., (*d*) 10 cm.

15. Find the length of sheet metal required for each of the following circular pipes, the diameters of which are given. Allow $\frac{5}{8}$″ extra for a seam.

(*a*) 6″, (*b*) 8″,
(*c*) 12″, (*d*) 1′ 3″,
(*e*) 7″, (*f*) 1′ $5\frac{1}{2}$″.



16. A piece of round stock 2 inches in diameter is being machined on a lathe.
   (*a*) What distance does a point on its circumference travel in 1 turn?
   (*b*) If the work revolves at a speed of 100 revolutions per minute (r.p.m.), what distance, in feet, does this point travel in 1 minute?

Review Exercise

B

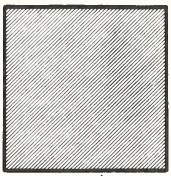
1. What is the perimeter of a square with each side 5 metres?
2. Find the cost of the fence for a small yard 50 feet by 60 feet if the fencing costs 14 cents per foot.
3. Find the circumference of a circle 14″ in diameter.
4. If a bicycle wheel is 26″ in diameter, how far will it roll in 200 turns if there is no slipping?
5. What is the length of one side of a square if its perimeter is 14′ 4″?
6. On a certain township plan a school yard is marked rectangular in shape and has dimensions of 5 chains and 3 chains 30 links. How many feet of fence would be needed to enclose it?
7. What is the length of tin necessary to make a cylindrical waste-basket if it has a circular cross-section 12 in. in diameter? Allow $\frac{1}{2}$″ for a seam.
8. What is the length of wire on a spool 3·5 inches in diameter if there is one layer of 60 turns?
9. In building a cottage it is customary to put a small drip board at the bottom of the siding and on top of the facing board, to keep the water from running under the facing board and causing rot. How many feet would a man order if his cottage was 24′ long and 20′ wide?
10. A flywheel 16″ in diameter is revolving at a rate of 280 r.p.m. Find the speed of a point on its rim in feet per minute.
11. Find the surface speed, in feet per minute, of an emery wheel which is turning at 600 r.p.m. if its diameter is 8·5″.
12. The circumference of a stove pipe is $25\frac{1}{8}$ inches. As stove pipes are sold according to their diameter what size of pipe would you order for this stove?
13. Which is the greater, and by how much, the perimeter of a hexagon 6″ to a side or the circumference of a circle of radius 6 inches?
14. A 75-foot fish line is to be wound on a spool with average diameter of $2\frac{1}{2}$ inches. How many turns are required?

CHAPTER 7

The Measurement of Area

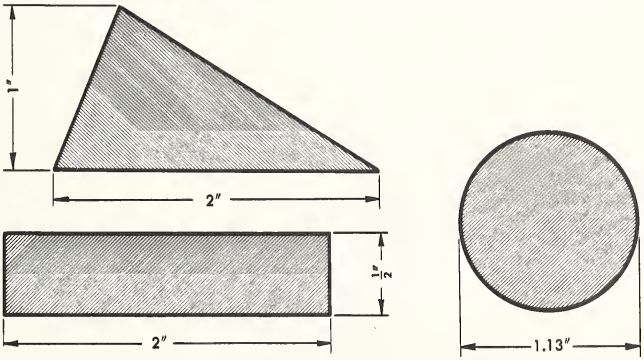
Units

A suitable unit for measuring small surfaces is a square inch. **A square inch is the area of surface contained in a square which has each side 1 inch long.**



1 square inch

One square inch of area may have any shape.



Each figure has an area of 1 square inch.

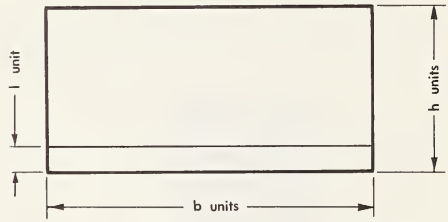
Other units for measuring the areas of surface are: the **square foot**, the **square yard**, the **square centimetre**, the **square metre** etc.

The Area of a Rectangle

The area of a rectangle may be obtained directly by applying a square inch to the rectangle but it is usually found by using the relationship between the measure of the area and the measures of its length and of its width.

In the chapter on linear measure we used letters to represent the measure of the length of a line in linear units. Similarly, measures of area may be represented by letters.

Let the area of the rectangle in square units be represented by A.
Let the length in corresponding linear units be represented by b.
Let the width in corresponding linear units be represented by h.



We are required to find a relationship between the measures A, b and h.

In a strip of the rectangle 1 linear unit deep, there are b square units of area.

There are h such strips.

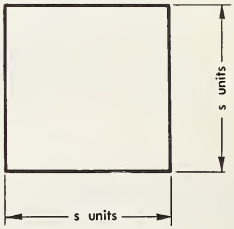
Therefore, the measure of the area in square units is bh.

But the measure of the area is also expressed in square units by A.

Therefore, $A = bh$.

The General Square

A square may be considered a rectangle in which the length and width are equal.



Let the length of each side in linear units be s.

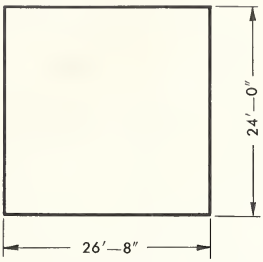
Then, using the relationship expressed by the formula, $A = bh$, the area of the square in square units is $s \times s$

$$= s^2.$$

$$\therefore A = s^2.$$

Example: Find the area of the floor of a room which is 26′ 8″ long and 24′ wide.

Solution:



The length of the floor of the room in feet is $26\frac{2}{3}$.
The width of the room in feet is 24.
The relationship for finding the area is, $A = bh$.
The area of the floor in square feet is $26\frac{2}{3} \times 24$
$= 640$.

Exercise

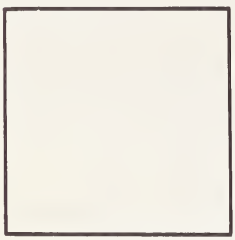
A

1. If the length of a rectangle is 12″ and its width is 6″, state its area in (*a*) square inches; (*b*) square feet; (*c*) square yards.
2. If the length of a rectangle is 2·5 cm. and its width is 2·0 cm., state its area in sq. cm. State the dimensions of this rectangle in mm. What is its area in sq. mm.?
3. A square has a side of 11 inches. What is its perimeter? What is its area?
4. State the square of each of the following numbers: (*a*) 9, (*b*) 12, (*c*) 13, (*d*) 16, (*e*) 17.
5. If a room is 22 feet long, 18 feet wide and 10 feet high, what is the area of its walls?
6. If the area of a rectangle is 3,612 sq. ft. and its width is 12 ft., what is its length?
7. State the value of $\frac{2{\cdot}4}{0{\cdot}8}$, $\frac{0{\cdot}35}{0{\cdot}5}$, $\frac{36{\cdot}}{1{\cdot}2}$, $\frac{1{\cdot}2}{0{\cdot}24}$.
8. What is the area of a square $\frac{1}{2}$″ to a side?
9. If the area of a square is given as follows, state the length of its side: (*a*) 25 sq. in., (*b*) $\frac{1}{16}$ sq. in., (*c*) 169 sq. cm., (*d*) $2\frac{1}{4}$ sq. m.

B

10. Make the necessary linear measurements and find the area of each of the following rectangles in suitable English square units.

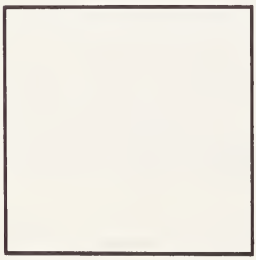
(a)



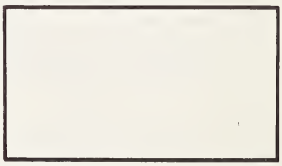
(b)

11. Make the necessary linear measurements and find the area of each of the following rectangles in suitable Metric square units.

(a)

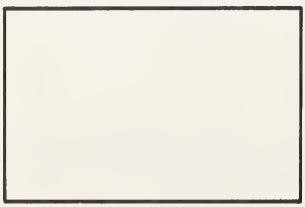


(b)

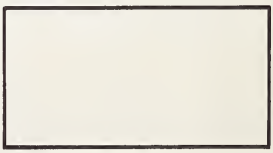


12. Make the necessary linear measurements and find the area of each of the following rectangles in (i) suitable English square units, (ii) suitable Metric square units.

(a)



(b)



13. Using compasses and ruler to make the right angle, construct accurately a rectangle $3\frac{1}{4}''$ long and $2\frac{1}{2}''$ wide. Calculate its area.

14. Using a protractor to make the right angle, construct accurately a rectangle 5·2 cm. long and 3·8 cm. wide. Find its area.

15. Construct 3 rectangles of different dimensions each having an area of 4 square inches.

16. Find the area of a rectangle 8′ 3″ long and 7′ 6″ wide.

17. Find the area of a rectangle 25 m. long and 156 cm. wide.

18. Find the area of the walls of a room that is 42 ft. long, 33 ft. wide and 9 ft. high. Assume there is no deduction for openings.

19. What will be the weight to the nearest pound of 15 sheets of No. 20 gauge galvanized sheet iron 24″ by 96″? From the handbook it is found that No. 20 gauge sheet iron weighs 1·65625 lb. per sq. ft.

20. Find the entire outside surface of a closed box which is $9\frac{1}{2}''$ long, 8″ wide and $6\frac{1}{2}''$ deep.

21. Find the outside surface of a cube each side of which is 1′ 6″.

22. Find the wall surface of a room 7·8 m. long, 6·5 m. wide and 3·2 m. high.

23. If a solid cube 6″ to a side has a slice 2″ in thickness removed, determine the outside area of the remaining part.

24. Find the length of one side of each square with area as follows:
(*a*) 225 sq. in., (*b*) 324 sq. cm., (*c*) 784 sq. ft., (*d*) 841 sq. yd.

C

25. Using the scale $3'' = 1'$ draw a square 1 foot to the side. Show all the square inches on this surface. State your conclusion as to the number of square inches in 1 square foot.

26. Draw a diagram to show that 1 square decimetre contains 100 square centimetres.

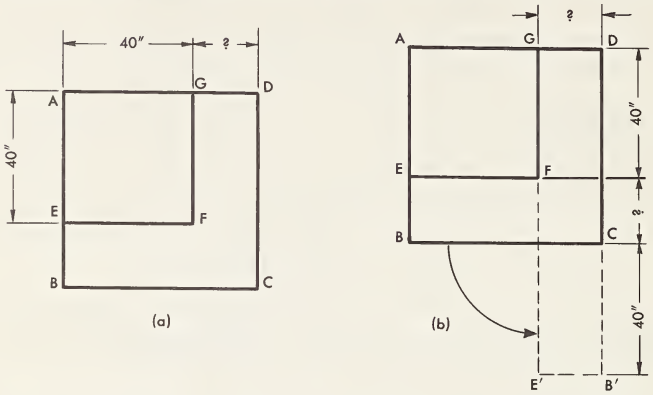
General Method of Finding Square Root

The factoring method of finding the square root of a number is useful only in the case of numbers which can be expressed readily as the product of two equal factors. Such numbers are relatively few. The following example will illustrate the basis of a general method of finding the square root of any number.

If the area of a square is 2,209 square inches, find the length of one side.

The length of one side in inches is found by taking the square root of 2,209.

Let **ABCD** represent a square the area of which is 2,209 square inches.



Our knowledge of the squares of integers will enable us to estimate that the length of one side is greater than 40 inches but less than 50 inches. Let us cut off a square **AEFG**, the length of one side of which is 40 inches. The part remaining is L-shaped and has an area in square inches of $2209 - 1600 = 609$.

Arrange the arms of the L-shaped figure to form a rectangle as in diagram (*b*). The length of this rectangle in inches is $2 \times 40 +$ the measure of the width

$= 80 +$ a number less than 10.

A reasonable estimate of the width in inches would be

$$\frac{609}{80 + \text{a number less than } 10} \text{ or } 7.$$

If we use 7 as the measure of the width, then the length of the rectangle in inches is $80 + 7$.

The area of the rectangle in square inches is 87×7

$= 609.$

But the area of this rectangle is known to be 609.

Evidently 7 is the correct measure of the width.

Therefore, the length of one side of the original square in inches is $40 + 7$

$= 47.$

Therefore $\sqrt{2209} = 47$.

Examine the following example which shows a mechanical method by which the square root of this same measure 2209 might be obtained. Compare each step with the corresponding step in the illustration above.

```
    47
4 | 22 | 09
  | 16
87| 609
  | 609
  |   0
```

$\sqrt{2209} = 47.$

1. Insert short lines to divide the number into two-digit groups beginning with the units digit.

2. The largest square less than the first group, 22, is 16. The square root of 16 is 4. Place the 4 above the 22 in the first space set aside for the square root.

3. Subtract the square of 4 from 22 and obtain 6. Bring down the next group of two digits to make 609.

4. Double the 4 (really 40) and place the 8 opposite the 609. Use 80 as a trial divisor to obtain the next digit 7.

5. Place the 7 with the 8 to make the divisor 87 and also place the 7 above the second group in the space beside the 4 to make 47.

6. Subtract the product 7×87 from 609. In this case the difference is zero and the operation is complete.

Example: Find the square root of (*a*) 729, (*b*) 98,596.

Solution: (*a*)

```
  2 7
2 | 7 | 29
  | 4
47| 329
  | 329
  |   0
```

$\sqrt{729} = 27.$

Note: In this case the first group contains one digit only.

(*b*)

```
    3 1 4
3 | 9 | 85 | 96
  | 9
61|  85
  |  61
624| 2496
   | 2496
   |    0
```

$\sqrt{98,596} = 314.$

Note: To find the third digit in the square root the first two digits 31 were doubled to give 62 (really 620) as a trial divisor. Then the process was continued as in the first example.

This process can be repeated as often as necessary to find the square root of numbers containing any number of digits.

Exercise B

1. If a square has an area of 1,369 square feet, what is the length of one side?

2. Find the square root of each of the following numbers:

(*a*) 5,041, (*b*) 1,849, (*c*) 7,569, (*d*) 1,444,
(*e*) 1,156, (*f*) 6,889, (*g*) 1,296, (*h*) 2,304,
(*i*) 4,096, (*j*) 1,521, (*k*) 5,184, (*l*) 4,900.

3. Find the square root of each of the following numbers:

(*a*) 529, (*b*) 784, (*c*) 676, (*d*) 625,
(*e*) 14,641, (*f*) 28,561, (*g*) 11,236, (*h*) 53,361,
(*i*) 822,649, (*j*) 654,481, (*k*) 649,636, (*l*) 405,769,
(*m*) 244,140,625, (*n*) 28,164,249.

4. Find the length of the side of a square equal in area to that of a rectangle with dimensions as follows:
   (*a*) 52 feet by 117 feet; (*b*) 55 feet by 220 feet.
5. What is the length of one side of a square which is equal in area to a rectangle the lengths of the sides of which are 99 feet and 275 feet respectively?
6. Find the length of the side of a square which is equal in area to the sum of two squares with sides
   (*a*) 25 ft. and 60 ft. respectively; (*b*) 28 ft. and 96 ft. respectively.
7. Find the length of the side of a square which is equal in area to the difference between the areas of two squares with sides of
   (*a*) 205 inches and 45 inches; (*b*) 305 cm. and 300 cm.

We have seen how the square roots of integers which are perfect squares may be calculated. The general method may be used to calculate the square roots of non-integral numbers which are perfect squares and also to find an approximate square root of any number.

Example 1: Find the square root of (*a*) 0·3364, (*b*) 82·2649.

For numbers containing digits to the right of the decimal point continue the division of the number into two-digit groups from the decimal point.

Solution: (*a*)

```
      0· 5 8
    5 | 0·33 | 64
      |  25  |
  108 |  864 |
      |  864 |
      |    0 |
```

$\sqrt{0{\cdot}3364} = 0{\cdot}58$

(*b*)

```
      9· 0 7
    9 | 82·26 | 49
      |  81   |
 1807 |  12649 |
      |  12649 |
      |      0 |
```

$\sqrt{82{\cdot}2649} = 9{\cdot}07$

Example 2: Find the square root of 20 correct to three significant digits.

Solution: Since the number 20, in this case, is considered to be exact we may affix zeros after the decimal point.

```
      4· 4 7 2
    4 | 20·00 | 00 | 00
      |  16   |
   84 |  400  |
      |  336  |
  887 |  6400 |
      |  6209 |
 8942 |  19100 |
      |  17884 |
      |   1216 |
```

The square root of 20 correct to 3 significant digits is 4·47.

Check: $4{\cdot}47 \times 4{\cdot}47 = 19{\cdot}9809$

It is evident that the square of 4·47 is very close to 20.

Exercise

A

1. State the square root of:

(*a*) 0·16, (*b*) 0·25, (*c*) 1·44,
(*d*) 0·09, (*e*) 0·0016, (*f*) 0·0009,
(*g*) 0·121, (*h*) 0·0081, (*i*) 0·000004,
(*j*) 0·000036, (*k*) 0·01, (*l*) 0·0001.

2. Find the first figure which is not zero in the square root of each of the following numbers:

(*a*) 0·8, (*b*) 0·09, (*c*) 0·70,
(*d*) 0·0005, (*e*) 1·25, (*f*) 0·1,
(*g*) 0·36, (*h*) 0·036, (*i*) 0·0049,
(*j*) 8·4, (*k*) 1,440, (*l*) 1·44.

B

3. If a square has an area of 31·36 square centimetres, what is the length of one side?

4. Find the square root of

(*a*) 8·41, (*b*) 0·1444, (*c*) 0·0961, (*d*) 0·005041.

5. Find, correct to the nearest tenth of an inch, the diagonal of a rectangle the length of which is 2 inches and the width of which is 3 inches. Consider these measures exact.

6. Find the square root of the following numbers correct to three significant digits:

(*a*) 1,000, (*b*) 0·5432, (*c*) 27.

7. Find correct to three decimal places

(*a*) $\sqrt{2}$, (*b*) $\sqrt{3}$, (*c*) $\sqrt{5}$.

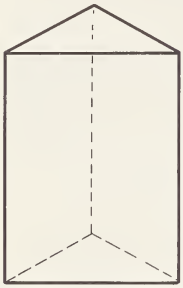
8. Express the measures under the square root sign in decimal notation and calculate the indicated square roots:

(*a*) $\sqrt{9\frac{3}{4}}$, (*b*) $\sqrt{2\frac{7}{16}}$, (*c*) $\sqrt{\frac{84}{3{\cdot}14}}$.

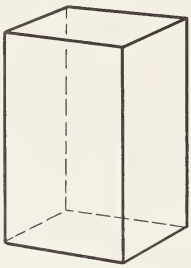
9. Find the side of a square which is equal in area to a rectangle 24 feet long and 72 feet wide.

Prisms

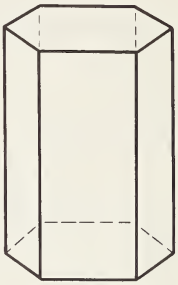
A solid which has rectangular sides that are perpendicular to the base and top is called a **prism**. A prism is classified according to the shape of its base.



Triangular Prism



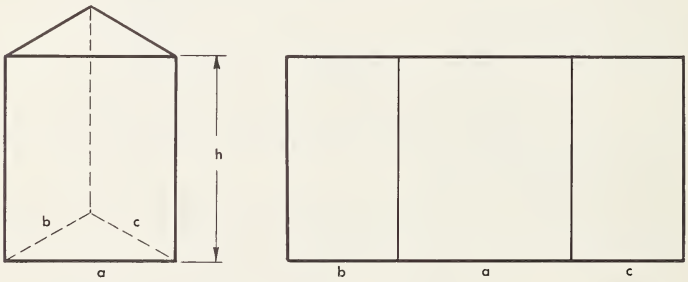
Rectangular Prism



Hexagonal Prism

Area of the Lateral Surface of a Prism

Suppose that it is required to find the lateral area of a triangular prism which has the dimensions as indicated on the diagram.



Let the area of the sides be A square units.

The sides could be arranged to form one long rectangle.

The length of the rectangle in linear units is $a + b + c$.

The width of the rectangle in linear units is h.

The area of the walls in square units is $h(a + b + c)$.

$$\therefore A = h(a + b + c).$$

Let the perimeter of the prism in linear units be p.

Then $$A = hp.$$

In a similar way it may be shown that the area of the lateral surface of any prism may be expressed by the relationship $A = ph$.

Example 1: Find the area of the walls of a room 35 feet long, 20 feet wide and 9 feet high.

Solution: Perimeter of the room in feet is $2(35 + 20) = 110$.

Relationship for area is $A = ph$.

Area of walls of room in square feet is $9 \times 110 = 990$.

Example 2: Find the area of the surfaces of the sides of a hexagonal prism if its height is 8·0 cm., and the lengths of the sides of the base 3·2 cm., 5·0 cm., 2·0 cm., 5·2 cm., 4·3 cm., and 2·7 cm. respectively.

Solution: Perimeter of base in cm. is $3{\cdot}2 + 5{\cdot}0 + 2{\cdot}0 + 5{\cdot}2 + 4{\cdot}3 + 2{\cdot}7$
$= 22{\cdot}4$.

Area of sides in sq. cm. is $8{\cdot}0 \times 22{\cdot}4$
$= 179{\cdot}2$
$= 180$ correct to 2 significant digits.

Exercise

A

1. Find the area of the walls of a triangular prism if the lengths of the sides of the triangular base are 3″, 4″ and 5″ respectively and its height is 10″.
2. Find the lateral area of a square prism, each side of the base of which is 2″ and the height of which is 5″.
3. Find the lateral area of an hexagonal prism, each side of the base of which is 1″ and the height of which is 6″.
4. A prism which is 8″ high has an equilateral triangle for a base. If the lateral area is 240 sq. in. find the length of each side of the equilateral triangle.
5. A prism which is 10″ high stands on a square base. The lateral area of the prism is 280 sq. in. Find the length of one side of the base.

B

6. Find the area of the sides of a square prism if the length of each side of the base and the height respectively are as follows:
   (*a*) 3″ and 10″; (*b*) 5·3 cm. and 4·2 cm.
7. Find the lateral area of a pentagonal prism if each side of its regular base is 4·36 cm. and its height is 8·23 cm.
8. Find the number of square feet of plywood needed to line the walls of a recreation room 24 feet long, 12 feet wide and 7 feet high.
9. Find the number of square feet of sheet metal needed to make a heating duct 18′ long, 14″ wide and 3″ deep.
10. Calculate the area of sheet metal required to make a heating duct 6′ long, 2′ 6″ wide and 12″ deep.

Cylinders

In common use are objects such as tin cans for food storage, pipes for carrying water or pistons for gasoline motors. Objects such as these are circular cylinders.

Courtesy Ewing Galloway

Grain Elevator

A cylinder may be considered a special case of a prism.

Example: Find the area of the curved surface of a cylinder 3·20 cm. in diameter and 4·30 cm. in length.

Solution:

The circumference of the base in cm. is $3{\cdot}14 \times 3{\cdot}20$

The area of the curved surface in sq. cm. is $4{\cdot}30 \times 3{\cdot}14 \times 3{\cdot}20$

$= 43{\cdot}2$ to 3 significant digits.

Exercise

A

1. If the circumference of a cylinder is 6·5″ and its height is 8″, what is the surface area?

2. If the circumference of a can of peas is 12″ and its height is 4″, what is the area of its walls?
3. If the diameter of a cylinder is 10″ and its height is 10″, what is the lateral area?

B

4. Find the area of the curved surface of a cylinder with a diameter of 2·25 inches and a height of 8·50 inches.
5. What will it cost to paint the walls of a circular tank 8 feet in diameter and 20 feet long at 50 cents a square foot? Give the cost to the nearest dollar.
6. A boiler has 80 tubes each 14 feet long and each with an inside diameter of 3·0″. Find the total heating surface.
7. How much cement enamel will be needed to paint the outside of a cement silo 12′ in diameter, 30′ high? Assume that 1 gallon covers 150 sq. ft.
8. What is the grinding surface on a stone 12″ in diameter and 2″ thick?

The Relationships between Different Square Units

The diagrams of the exercise, page 131, part C, should help you to understand how a square unit is related to other square units. Since 1 foot = 12 inches, therefore 1 square foot = 12 × 12 square inches; also 10 cm. = 1 dm., therefore 10 × 10 sq. cm. = 1 sq. dm. etc. Thus we see how the square measure relationships can be built up from the linear measure relationships. The acre and the hectare are, however, special area units used to measure land. The acre was originally defined as a piece of ground 40 rods long and 4 rods wide. The hectare is 1 square hectometre.

English Square Measure Relationships

144 square inches (sq. in.)	= 1 square foot (sq. ft.)
9 square feet	= 1 square yard (sq. yd.)
$30\frac{1}{4}$ square yards	= 1 square rod (sq. rd.)
160 square rods	= 1 acre (ac.)
640 acres	= 1 square mile (sq. mi.)
10 square chains (sq. ch.)	= 1 acre.

Metric Square Measure Relationships

100 square millimetres (sq. mm.)	= 1 square centimetre (sq. cm.)
100 square centimetres	= 1 square decimetre (sq. dm.)
100 square decimetres	= 1 square metre (sq. m.)
10,000 square metres	= 1 hectare.

A hectare is approximately 2·47 acres.

Exercise

A

1. State the number of square inches in
   (*a*) $\frac{1}{2}$ square foot, (*b*) $\frac{1}{3}$ square foot, (*c*) 2 square feet,
   (*d*) 1 square yard, (*e*) 0·1 square foot, (*f*) 0·05 square foot.
2. How many square rods in a square $5\frac{1}{2}$ yards to a side?
3. Complete the following statements:
   (*a*) 1 cm. = ... mm. (*b*) 1 sq. cm. = ... sq. mm.
   (*c*) 1 dm. = ... mm. (*d*) 1 sq. dm. = ... sq. mm.
   (*e*) 1 m. = ... cm. (*f*) 1 sq. m. = ... sq. cm.
   (*g*) 0·345 sq. m. = ... sq. cm. (*h*) 240 sq. cm. = ... sq. m.
4. A square has an area of 1 square mile. How many yards are in the length of one side?
5. For each rectangular area suggest the linear dimensions of at least three rectangles.

Area		*Rectangle No. 1*		*Rectangle No. 2*		*Rectangle No. 3*	
units measure		*length*	*width*	*length*	*width*	*length*	*width*
sq. in.	12						
sq. ft.	15						
sq. cm.	36						
sq. mm.	48						

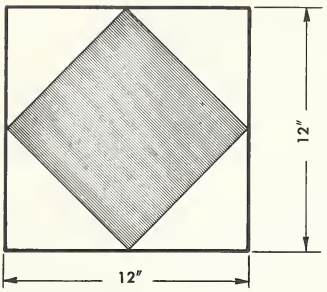
6. State the dimensions of three rectangles, each of which would contain 1 acre.
7. If 1 inch = 2·54 centimetres, how would you calculate the number of square centimetres in 1 square inch?

B

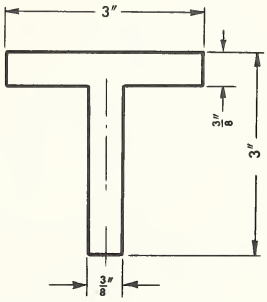
8. Calculate the number of square centimetres in 1 square inch correct to three significant digits.
9. How many square yards are there in 1 acre?
10. A farmer says a field is 60 rods long and 40 rods wide. How many acres does it contain?
11. A piece of cork 24″ by 36″ is worth 96 cents. At this same rate, how much would you pay for a piece 12″ by 15″?
12. Find the area in acres of a plot of ground 160 rods long and 420 rods wide.

13. A man wishes to buy an acre of ground rectangular in shape. He can buy property to a depth of 550 feet. How many feet of frontage should he purchase?

14. Find the area of the shaded portion of this square formed by joining the middle points of the sides.



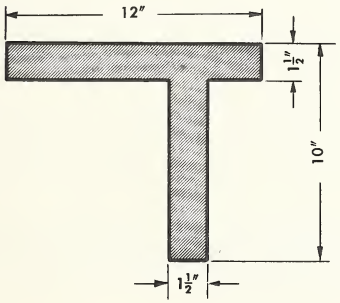
15. Find the area of the T-shaped object.



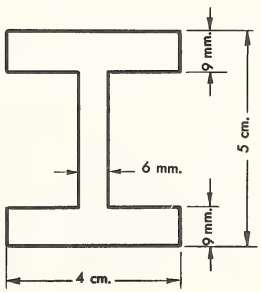
16. A rectangle has an area of 121 sq. yd. and a width of 33 ft. Find the length of the rectangle in feet.

17. Find the area in acres of a school yard 99 yards long and 88 yards wide.

18. The shaded T is cut from a piece of sheet metal. Find the area of the waste.



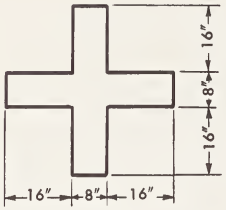
19. Find the area of the figure illustrated.



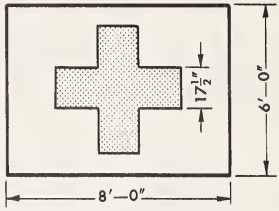
20. At 25 cents a square metre, find the cost of painting one side of a solid board fence which is 2·6 cm. high and 76·8 m. long.

21. At $2 per square yard, what will it cost to put linoleum on the treads of the steps and the landing of the stairway if there are 10 steps 3 feet long and 10 inches wide and the landing measures 10 feet long and 6 feet wide?

22. Find the area of the cross.



23. Find the area of the white portion of the red-cross flag.



24. If, in a theatre, a person occupies a floor space 27″ by 33″, find the area required to seat 600 people. Allow 10% extra for aisles etc.

25. How many wooden blocks 3″ by 4″ would be required for a shop floor 18′ by 12′ 6″?

26. Find the area of the walls of a room 18′ long, 12′ wide and 10′ high if there are two doors 7′ by 3′ 6″ and 4 windows 3′ by 22″.

27. What is the area in square metres of a rectangle the length of which is 1 metre and the width, 5 decimetres?

28. A chemical tank is to be lined on the sides and bottom with a thin sheet of lead. If the tank is 65 cm. long, 28 cm. wide and 19 cm. high, find the area of the lead surface.

29. Find the length of one side of a square which has an area of
(*a*) 5,184 sq. in., (*b*) 35 sq. ft., (*c*) 2 sq. cm.

C

30. Find the cost of asphalt shingles for a roof 24 feet by 14 feet at $9 per square. A square of shingles consists of 3 bundles and will cover 100 square feet. Fractional parts of bundles cannot be purchased.

31. Find the cost of putting insul-brick siding on the walls of a cottage 24′ long, 20′ wide and 8′ to the roof. It has 150 sq. ft. of window and door space. Shingles cost $16.50 per square and may be purchased in ½ square lots. Allow ½ square for waste.

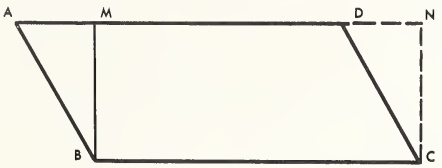
32. A rectangle is 20 inches long and 10 inches wide. Calculate the area in square centimetres. Since the area is required in square centimetres, first express the linear dimensions in centimetres.

Areas of Other Figures

Since the adjacent sides of a rectangle are at right angles the area of a rectangle can be found directly or can be calculated. The area of any other figure is calculated by first relating it to a rectangle the dimensions of which can be found.

Area of a Parallelogram

Consider the parallelogram **ABCD**. If a straight line were drawn through **B** perpendicular to **AD**, it would cut off a small right-angled triangle **ABM**. Now, if this triangle were moved to the opposite end of the parallelogram, the line **AB** would fit exactly along the line **CD** and the parallelogram would be converted into a rectangle **MBCN**. This rectangle would have the same area as the parallelogram **ABCD**.



The base **BC** of the rectangle is the same as the base of the parallelogram. The height **BM** of the rectangle is the same as the altitude of the parallelogram.

Let the area of the parallelogram in square units be A.

Let the length of the base of the parallelogram in linear units be b.

Let the altitude of the parallelogram in linear units be h.

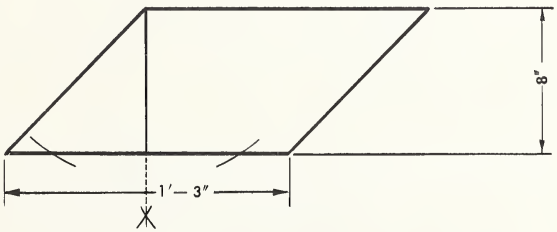
The area of the rectangle **MBCN** in square units is bh.

Therefore the area of the parallelogram in square units is bh.

$$\therefore A = bh.$$

Example: Find the area of a parallelogram with a base of 1′ 3″ and an altitude of 8″.

Solution:



The base of the parallelogram in feet is $1\frac{1}{4}$.

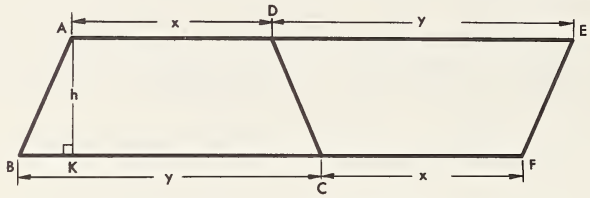
The altitude of the parallelogram in feet is $\frac{2}{3}$.

The relationship for finding the area is $A = bh$.

The area of the parallelogram in square feet is $\frac{5}{4} \times \frac{2}{3} = \frac{5}{6}$.

Area of a Trapezoid

A trapezoid is a quadrilateral in which two sides are parallel.



Let **ABCD** be a trapezoid with **AD** parallel to **BC**.

AK is drawn perpendicular to **BC**.

Let the length of **BC** in linear units be y.

Let the length of **AD** in linear units be x.

Let the altitude **AK** in linear units be h.

Let the area of the trapezoid in square units be A.

Extend **AD** to **E** so that **DE** is equal to **BC** and extend **BC** to **F** so that **CF** = **AD**.

DCFE is a trapezoid exactly the same as **ABCD** and **ABFE** is a parallelogram with base $(x + y)$ units and height h units.

$\therefore$ area of trapezoid $= \frac{1}{2}$ area of corresponding parallelogram

$$\therefore A = \tfrac{1}{2}h(x + y).$$

A

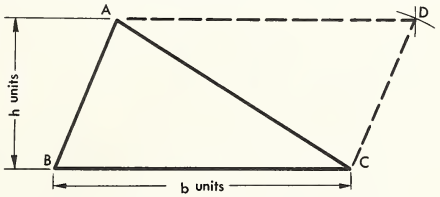
1. State the area of the following parallelograms:
   (*a*) base 4 ft., altitude 3 ft.; (*b*) base 12 cm., altitude 7 cm.;
   (*c*) base $1\frac{1}{2}$ yd., altitude 7 yd.; (*d*) base 40 m., altitude 35 m.
2. If a parallelogram has an area of 36 sq. in. and the lengths of the sides are 9″ and 6″ determine
   (*a*) the perpendicular distance between the 9″ sides;
   (*b*) the perpendicular distance between the 6″ sides.
3. If a parallelogram has sides of 6″ and 4″ and the perpendicular distance between the 6″ sides is 3″, determine the distance between the 4″ sides.
4. The lengths of the two parallel sides of a trapezoid are 4″ and 6″ respectively and the distance between them is 10″. Find the area of the trapezoid.
5. The area of a trapezoid is 35 sq. ft. and the lengths of the parallel sides are 4 ft. and 3 ft. respectively. Find the distance between the parallel sides.

B

6. Find the area of a parallelogram with a base of 3·5 cm. and an altitude of 76 mm.
7. The base of a parallelogram is 10·2″ and its altitude is 7·45″. Find its area.
8. A damper plate in the form of a parallelogram has an area of 120 sq. in. If the perpendicular distance between a pair of parallel sides is 8″, find the length of these sides.
9. A parallelogram is to have an area of 1,024 square inches. The adjacent sides are 64 inches and 32 inches respectively.
   Find: (*a*) the altitude to the 64-inch side,
   (*b*) the altitude to the 32-inch side.
10. Find the area of a trapezoid with parallel sides 10·2″ and 8·7″ respectively and altitude 5·4″.
11. A trapezoid with an altitude of 15 yards has an area of 240 square yards. Find the length of the base if the opposite parallel side is 11 yards.
12. A trapezoidal-shaped field is 50 rd. wide and the lengths of the parallel sides are 30 rd. and 60 rd. respectively. Find its area in acres.

Area of a Triangle

We might deduce the relationship for the area of a triangle by comparing it with a parallelogram having the same base and the same altitude.



If we use the sides **AB** and **BC** of the triangle **ABC** and complete the parallelogram **ABCD**, we notice that the area of the parallelogram is twice the area of the triangle.

Let the area of the triangle in square units be A.

Let the base of the triangle in linear units be b.

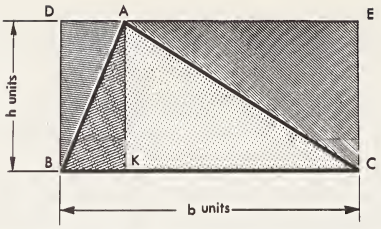
Let the altitude of the triangle in linear units be h.

Then the area of the corresponding parallelogram in square units is bh.

Therefore, the area of the triangle in square units is $\frac{1}{2}bh$.

$$\therefore A = \tfrac{1}{2}bh.$$

We might deduce this relationship by comparing the triangle with a rectangle having the same base and the same altitude.

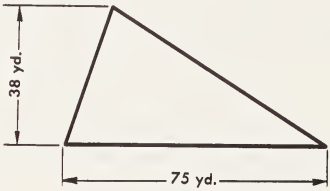


In the triangle **ABC**, **AK** is drawn perpendicular to **BC**. The rectangle **DBCE** and the triangle **ABC** have the same base **BC** and the same height **AK**.

With this information, show how the formula $A = \frac{1}{2}bh$ can be developed.

Example: A triangular piece of land measures 75 yards along one side and the perpendicular distance from the opposite vertex to this side is 38 yards. Find its area.

Solution:



The base of the triangle in yards is 75.

The altitude of the triangle in yards is 38.

The relationship for finding the area is $A = \frac{1}{2}bh$.

The area of the triangle in square yards is $\frac{1}{2} \times 75 \times 38$

$= 75 \times 19$

$= 1{,}425.$

Exercise

A

1. If the base of a triangle is 6·0 cm. and the altitude from the opposite vertex to this base is 8·0 cm., what is the area of the triangle?
2. An equilateral triangle with a side of 10″ has a height of 8·7″. What is its area?
3. An isosceles triangle which is 12″ high has a base of 10″. What is its area?
4. A right-angled triangle has adjacent sides about the right angle of 18″ and 20″. What is the area of the triangle?

5. Find the value of

(*a*) $\frac{1}{2} \times 2{\cdot}3 \times 18$, (*b*) $\frac{1}{2} \times 0{\cdot}42 \times 7$, (*c*) $\frac{1}{2} \times 8^2$,

(*d*) $\frac{1{\cdot}2 \times 1{\cdot}3}{2}$, (*e*) $\frac{0{\cdot}04 \times 0{\cdot}06}{2}$, (*f*) $\frac{12{\cdot}4 \times 0{\cdot}1}{2}$.

6. If a triangle has an area of 20 sq. in. and its base is 4″, how far is the opposite vertex from this base?

7. If the area of a triangle and the length of its base are as follows, determine the height of the triangle:

(*a*) 48 sq. cm., 24 cm.; (*b*) 100 sq. cm., 50 cm.;

(*c*) 1 sq. ft., 6 in.; (*d*) 1 sq. yd., 1 ft.

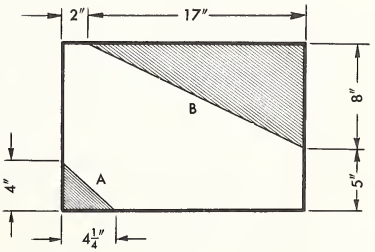
B

8. Find the area of a triangle if the base is 12 feet and the height or altitude is 16 feet.

9. Find the area of a triangle if the base is 8·25 ft. and the height is 7·78 ft.

10. A triangle has a base of 400 yd. and a perpendicular height of 300 feet. Find its area in square yards.

11. An irregular piece of sheet metal is made from a rectangle by cutting off the corners **A** and **B** as shown in the diagram. Find the area of the piece so formed.



12. Find the area in acres of a triangular field if one side is 40 rods and the perpendicular to this side from the opposite vertex is 28·2 rods.

13. A triangular crest measures 81 mm. on one side and is 78 mm. in altitude. Find its area in square centimetres.

14. Construct a triangle with a base of 3″ and equal sides of $2\frac{1}{4}$″. With ruler and compasses construct the perpendicular from the vertex to the 3″ side. Measure the length of the perpendicular and calculate the area of the triangle.

15. Construct a triangle with sides of $1\frac{1}{4}$″, 3″ and $3\frac{1}{4}$″.

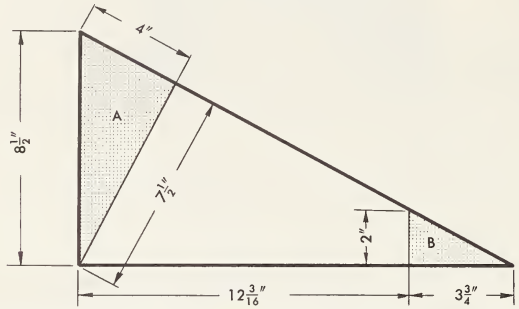
(*a*) Measure and record the size of the largest angle.

(*b*) Calculate the area of this special triangle.

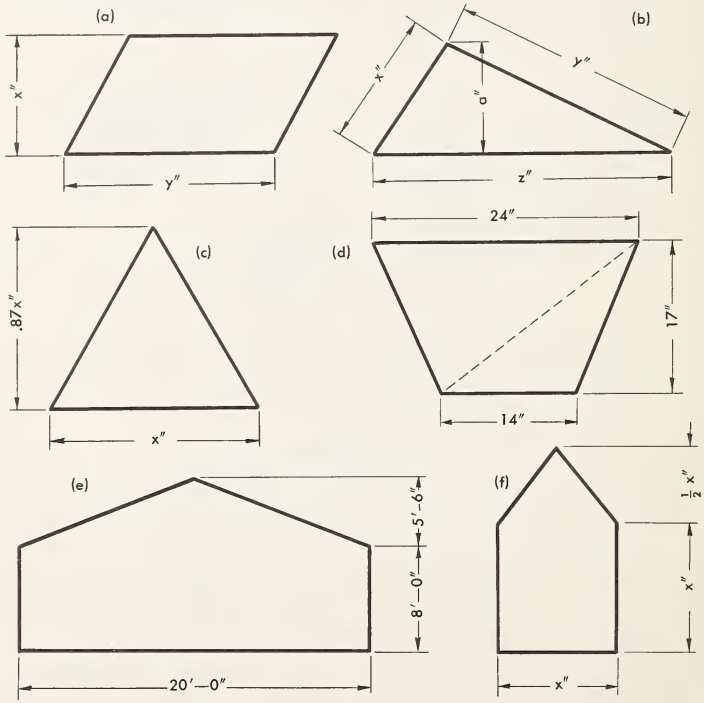
(*c*) Using $3\frac{1}{4}$″ as the base draw the perpendicular from the opposite vertex. Measure this perpendicular and calculate the area of the triangle.

C

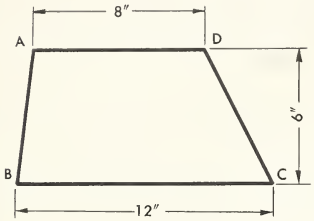
16. When triangles **A** and **B** are cut from a large triangular piece as shown, find the area of the remaining quadrilateral.



17. In terms of the measures given, find the areas of the figures illustrated by the following diagrams:



18. In an irregular quadrilateral **ABCD**, the diagonal **BD** is 5″ in length. The perpendicular drawn from **A** to **BD** is $3\frac{1}{4}$″ and the perpendicular from **C** to **BD** is $3\frac{7}{8}$″. Calculate the area of this quadrilateral.



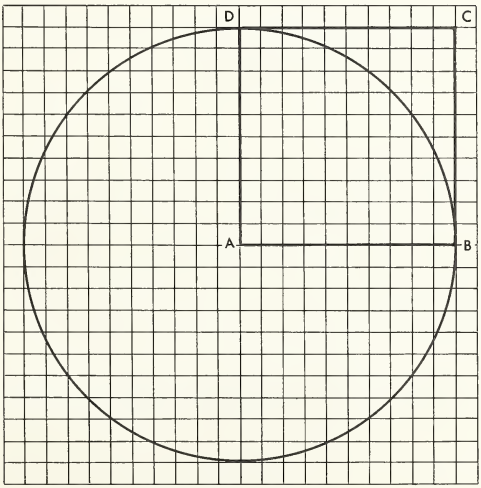
19. **ABCD** is a quadrilateral in which **AD** is parallel to **BC**. From the dimensions given on the diagram, calculate the area.

Area of a Circle

The following experiments will illustrate how to find a *rectangle* to which the area of a circle can be related:

Experiment 1:

Using as centre a main intersection on a sheet of graph paper, construct a circle of radius 1 inch. Find, by counting, the number of small squares inside the circumference. If a part of a small square is less than $\frac{1}{2}$ of a small square, neglect it; if a part square is larger than $\frac{1}{2}$ a square, count it as a full square.



Using the lines of the squared paper, construct a square **ABCD** which has the radius of the circle as one side. Count the number of small squares in **ABCD**. Compare by division the measure of the area of the circle with that of the square.

Record your results in a table similar to the one below. Repeat the experiment, using radii of different lengths such as $\frac{1}{2}''$, $1{\cdot}2''$, $1{\cdot}5''$, $1{\cdot}7''$ and $2''$. Complete the table.

Measure of the area of the circle in small squares	*Measure of the area of square ABCD in small squares*	$\dfrac{\text{Area of circle}}{\text{Area of square on radius}}$

If your work has been reasonably accurate you will discover from your records that the measure of the area of a circle is just over three times the measure of the area of the square. By other methods this number has been found to be the same as the number we used in the relationship between the circumference and diameter of a circle. It is represented by π. Therefore, the measure of the area of a circle is π times the measure of the area of the square on the radius. This relationship is expressed by the formula

$$A = \pi r^2.$$

Experiment 2:

The area of a circle can be related to the area of a rectangle by cutting the circle into a great number of sectors. These pieces are then separated and placed together again in a form to approximate a rectangle. The more pieces we have the more nearly the figure approximates a rectangle.

The measure of the length of this rectangle is approximately the same as the measure of $\frac{1}{2}$ the circumference.

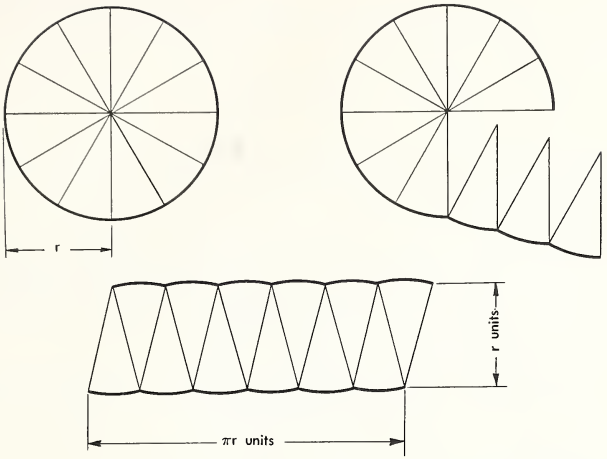
The width of this rectangle is approximately the same as the radius of the circle.

Then the measure of the area of the rectangle in square units is $\frac{1}{2}$ the measure of the circumference multiplied by the measure of the radius of the circle.

Let the measure of the area of the circle in square units be A.

Let the measure of the circumference in linear units be C.

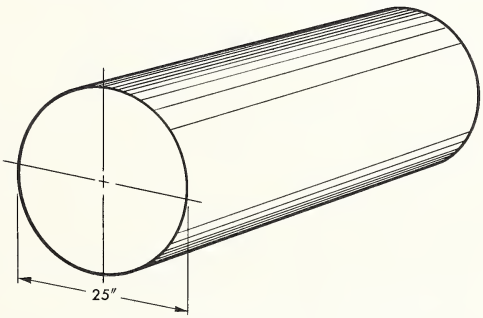
Let the measure of the radius in linear units be r.



Then

$$A = \tfrac{1}{2}C \times r$$
$$= \tfrac{1}{2} \times 2\pi r \times r$$
$$\therefore A = \pi r^2.$$

Example: A large steel drum has a diameter of 25″. Find the area of one end.
Solution:



The diameter of the circle in inches is 25.
The radius of the circle in inches is 12·5.
The relationship for finding the area is $A = \pi r^2$.
The area of the circle in square inches is

$$3{\cdot}142 \times 12{\cdot}5 \times 12{\cdot}5$$
$$= 490{\cdot}9 \text{ or } 491 \text{ to 3 significant digits.}$$

In this problem the data were considered exact and the value of π was taken as 3·142 in order that the final result would be reliable to at least 3 significant digits.

Exercise

A

1. State the expression which, if simplified, would give the area of each of the following circles:
   (*a*) radius 10″, (*b*) diameter 24″, (*c*) radius $6\frac{1}{2}$″,
   (*d*) radius $\frac{1}{2}$″, (*e*) diameter $\frac{3}{4}$″, (*f*) diameter 4,000 miles,
   (*g*) radius 1·3 cm., (*h*) radius 0·05 cm., (*i*) diameter 35·5 m.

2. State the expression which, if simplified, would give the circumference of each of the following circles:
   (*a*) radius 10″, (*b*) diameter 22″, (*c*) diameter 16·4 cm.,
   (*d*) radius 7·5′, (*e*) diameter 4,000 miles, (*f*) radius 3·56 mm.

3. Explain how you would find the diameter of a circle if you knew its circumference.

4. Explain how you would find the measure of the radius of a circle if you knew its area.

B

5. Calculate the area of each of the following circles:
   (*a*) radius is $3\frac{1}{2}$″, (*b*) diameter is 5″,
   (*c*) radius is 64″, (*d*) diameter is 21·8″.

6. Find the area of a circle having a radius 14″.

7. Find the cross-sectional area of an engine cylinder the diameter of which is 5·6″.

8. Find the cost of sodding a circular plot of ground 40 yd. in diameter at 50¢ a square yard.

9. Find the area of the waste material when the largest possible circle is cut from a 7″ square of sheet metal.

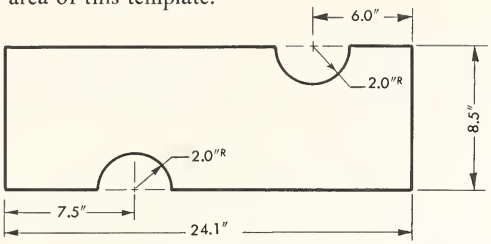
10. A piston is subjected to a pressure of 36 lb. per sq. in. If it has a diameter of 3·3 in., find the total pressure.

11. Find the area of a cross-section of a steel rod $\frac{7}{8}$″ in diameter.

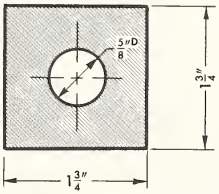
12. Which has the greater area, a circle 10″ in diameter or a square 9″ to a side? What is the difference in their areas?

13. How many times is the area of a 6″ circle that of a 2″ circle?

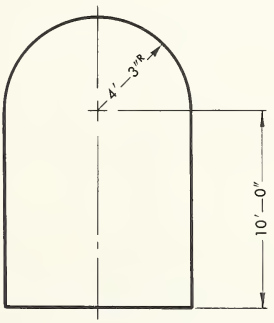
14. Find the area of this template.



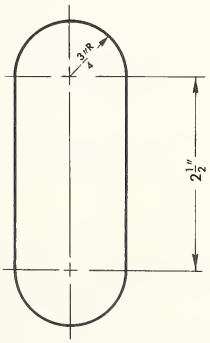
15. Find the area of the shaded part of this washer.



16. Find the area of the church window with the shape and dimensions as shown.



17. Find the area of this cover plate for an elbow on an electrical conduit pipe.



18. (*a*) If the diameter of a circle is 3″, find the radius, the circumference and the area of the circle.

(*b*) If the radius of a circle is 4″, find the diameter, the circumference and the area of the circle.

(*c*) If the diameter of a circle is 0·120″, find the radius, the circumference and the area of the circle.

(*d*) If the radius of a circle is 0·35″, find the diameter, the circumference and the area of the circle.

19. (*a*) If the circumference of a circle is 31·4″, find the diameter, the radius and the area of the circle.
(*b*) If the circumference of a circle is 235 cm., find the diameter, the radius and the area of the circle.
(*c*) If the circumference of a circle is 44″, find the diameter, the radius and the area of the circle.

20. (*a*) If the area of a circle is 314 sq. ft., find the radius, the diameter and the circumference of the circle.
(*b*) If the area of a circle is 78·5 sq. in., find the radius, the diameter and the circumference of the circle.
(*c*) If the area of a circle is 113·04 sq. cm., find the radius, the diameter and the circumference of the circle.

C

21. Compare by division the area of an 8″ circle with that of the area of the square on its diameter.

22. On squared paper draw two circles in which the diameter of one is twice the diameter of the other. Compare by division the area of the larger circle with that of the smaller circle.

23. If a semicircle is 18·6 cm. in diameter, determine
(*a*) the perimeter of the semicircle, (*b*) the area of the semicircle.

24. A circle 20·4 cm. in diameter is divided into quadrants. What is the perimeter and the area of one of these quadrants?

25. Find the radius of each of the circles with area as follows:
(*a*) 28·26 sq. in., (*b*) 62·8 sq. ft., (*c*) 15·7 sq. cm.

26. What is the length of the arc subtended by an angle of 30° at the centre of an 8″ circle?

Right-Angled Triangle Relationship

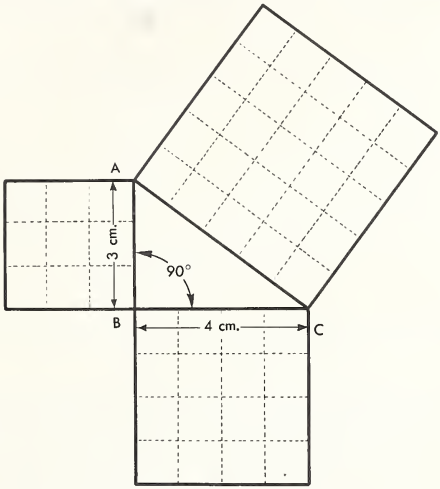
Instead of walking round the two sides of a rectangular lot, sometimes we are tempted to cut across the corner. Do we save much distance by using the so-called short cut? This problem, like many others that frequently arise in science and industry, can be solved by using a special relationship which exists among the lengths of the sides of a right-angled triangle.

It will be recalled that a right-angled triangle is one in which one angle is a right angle. The side of the triangle opposite the right angle is called the **hypotenuse**. The other two sides are usually referred to as the sides about the right angle.

Try to discover the special relationship referred to above as you work the following exercise.

Exercise B

1. Draw a right-angled triangle **ABC** in which $\angle \mathbf{B} = 90°$, **AB** = 3 cm. and **BC** = 4 cm. Measure the length of the hypotenuse. On **AB**, **BC** and **CA** respectively, construct squares as shown in the diagram.



Find the area of each of these squares. Compare the sum of the areas of the squares on **AB** and **BC** with that of the square on **AC**.

2. Draw a triangle **DEF** in which $\angle \mathbf{E} = 90°$, **DE** = 2·5 cm. and **EF** = 6 cm. Measure the length of the hypotenuse. Construct a square on each of the three sides. Compare the area of the square on the hypotenuse with the sum of the areas of the squares on the sides about the right angle.

3. Draw a triangle **ABC** in which $\angle \mathbf{B} = 120°$, **AB** = 2 cm. and **BC** = 3 cm. Measure the longest side, **AC**. Construct a square on each of the three sides. Compare the area of the square on the longest side with the sum of the areas of the squares on the other two sides.

4. Construct a triangle **KMN** in which **KM** = 3 cm., $\angle \mathbf{M} = 70°$ and **MN** = 4 cm. Measure the length of **KN**. Compare the area of the square on the longest side of this triangle with the sum of the areas of the squares on the other two sides.

5. Classify as right-angled, obtuse-angled or acute-angled each of the triangles with lengths of sides as follows:

(*a*) 5″, 12″ and 13″; (*b*) 2″, 3″ and 4″;
(*c*) 7″, 8″ and 9″; (*d*) 20″, 21″ and 29″;
(*e*) 25″, 24″ and 7″; (*f*) 24″, 70″ and 74″.

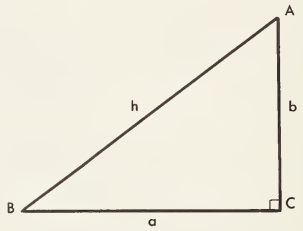
These comparisons should help you to draw the following conclusion which is true for right-angled triangles only:

In any right-angled triangle the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the sides about the right angle.

This principle is often called the Pythagorean relationship, in honour of Pythagoras (582–507 B.C.) who first proved it to be true for all right-angled triangles.

The Pythagorean relationship may be expressed by a formula as follows:

Let the length of the hypotenuse in linear units be h.
Let the lengths of the other two sides in linear units be a and b respectively.
Then

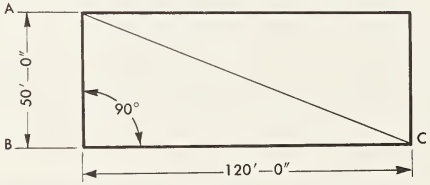


$$h^2 = a^2 + b^2.$$

We will use this relationship to calculate the length of the third side of a right-angled triangle when the lengths of the other two sides are known.

Example 1: A vacant corner lot is 50 feet wide and 120 feet deep. What is the distance from a corner to the diagonally opposite corner?

Solution:



From the diagram we see that **AC**, the diagonal of the lot, is the hypotenuse of the right-angled triangle **ABC**.

The area of the square on **AB** in square feet is $50 \times 50 = 2{,}500$.

The area of the square on **BC** in square feet is $120 \times 120 = 14{,}400$.

By the Pythagorean relationship, the area of the square on **AC** in square feet is $2{,}500 + 14{,}400 = 16{,}900$.

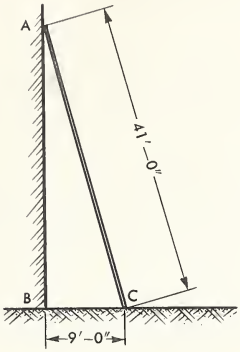
The length of **AC** in feet is $\sqrt{16{,}900}$

$$= \sqrt{169 \times 100}$$
$$= 13 \times 10$$
$$= 130.$$

The distance from corner to corner in feet is 130.

Example 2: A ladder 41 feet in length is placed against a wall with its foot 9 feet from the base of the wall. To what height on the wall does the ladder reach?

Solution:



The ladder **AC** completes the right-angled triangle with the wall **AB** and the ground **BC**.

The area of the square on **AC** in sq. ft. is $41 \times 41 = 1{,}681$.

The area of the square on **BC** in sq. ft. is $9 \times 9 = 81$.

The difference in the areas of the squares on **AC** and **BC** in square feet is 1,600.

Therefore, **AB** in feet is $\sqrt{1{,}600}$

$= 40.$

The height in feet is 40.

Example 3: Find the length of the hypotenuse of a right-angled triangle if the lengths of the sides are 0·83″ and 0·45″ respectively.

Solution: Let the length of the hypotenuse in inches be h.

The right-angled triangle relationship is expressed by

$$h = \sqrt{a^2 + b^2}$$
$$\therefore h = \sqrt{0{\cdot}83^2 + 0{\cdot}45^2}$$
$$= \sqrt{0{\cdot}8914}$$
$$= 0{\cdot}94.$$

The length of the hypotenuse is 0·94 inches.

Exercise

A

1. Find the length of the hypotenuse of a right-angled triangle if the lengths of the sides are as follows:

(*a*) 3 cm. and 4 cm. respectively;

(*b*) 5 inches and 12 inches respectively.

2. Find the length of the third side of a right-angled triangle if the hypotenuse and one of its sides are as follows:

(*a*) 10 cm. and 8 cm. respectively;
(*b*) 15 miles and 9 miles respectively.

B

3. In triangle **ABC**, $\angle$**B** is 90°. Find

(*a*) **AC** when **BC** = 12″ and **AB** = 16″;
(*b*) **BC** when **AC** = 13″ and **AB** = 12″;
(*c*) **AB** when **AC** = 17″ and **BC** = 15″.

4. If the lengths of the sides about the right angle of a right-angled triangle are 9 feet and 40 feet respectively, what is the length of the hypotenuse?

5. If a vacant corner lot is 72′ by 30′, what distance is saved by taking a short cut from corner to corner?

6. Is it possible for John to put his fishing pole, which is 35″ long, in the bottom of his suitcase which is 30″ long and 16″ wide?

7. A ladder 25 feet long just reaches a window when the foot of the ladder is 7 feet from the wall. How high is the window above the ground?

8. If a rectangle is 60 feet long and 25 feet wide, find

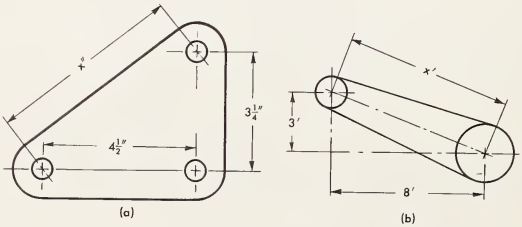
(*a*) the perimeter, (*b*) the area, (*c*) the length of each diagonal.

9. Calculate the length of the hypotenuse of each of the right-angled triangles if the lengths of the other two sides are as follows:

(*a*) 24 in. and 7 in. respectively;
(*b*) 10·5 cm. and 15·0 cm. respectively;
(*c*) 0·75 cm. and 1 cm. respectively;
(*d*) 8·0 in. and 9·0 in. respectively;
(*e*) 12·0 in. and 12·0 in. respectively.

10. Find the length of the third side of each right-angled triangle if the lengths of the hypotenuse and one of the sides respectively are as follows:

(*a*) 41 ft. and 9 ft.; (*b*) 6·5 mi. and 1·1 mi.;
(*c*) 2·00 cm. and 1·00 cm.; (*d*) 35·0 ft. and 25·0 ft.



11. Calculate the missing dimension in each of the diagrams above.

Review Exercise

A

1. Give the dimensions of at least three rectangles which would contain 1 square inch of area.

2. What is the area of a rectangle which is 6 inches long and 8 inches wide?

3. If the area of a rectangle is 72 square centimetres and its width is 4 centimetres, what is its length?

4. (*a*) How many square inches are in 2 square feet?
(*b*) How many square yards are in 189 square feet?
(*c*) How many square yards are in 2 square rods?
(*d*) How many square centimetres are in 1 square metre?
(*e*) How many square millimetres are in 8 square centimetres?
(*f*) How many square chains are in 5 acres?
(*g*) How many square rods are in 1 square chain?

5. Define a parallelogram.

6. State the formula for
(*a*) the area of a rectangle, (*b*) the area of a square,
(*c*) the area of a parallelogram, (*d*) the area of a circle,
(*e*) the area of a triangle, (*f*) the area of a trapezoid.

7. State the length of the base and the altitude of a triangle which contains 20 square inches.

8. A right-angled triangle has sides of 6″, 8″ and 10″. Determine its area.

9. A right-angled triangle has sides of 5″, 12″ and 13″. Determine its area.

10. What is the first digit in the square root of each of the following numbers:
(*a*) 0·546, (*b*) 3·05, (*c*) 187, (*d*) 189,980.

B

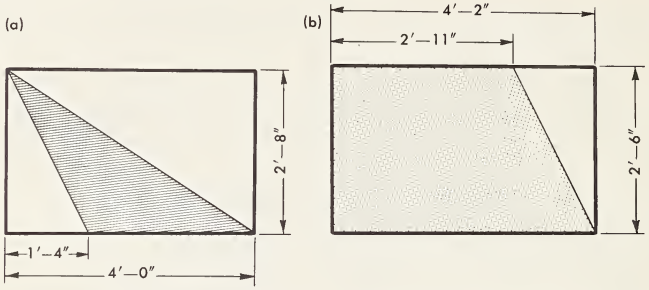
11. Copy and complete each of the following:

(*a*) 3 ac.	= ... sq. yd.	(*b*) 1 sq. mi.	= ... ac.
(*c*) 5 ac.	= ... sq. ch.	(*d*) 10 sq. rd.	= ... sq. yd.
(*e*) 3 sq. ch.	= ... sq. yd.	(*f*) 8 sq. yd.	= ... sq. ft.
(*g*) 10 sq. ft.	= ... sq. in.	(*h*) 5,000 sq. in.	= ... sq. ft.

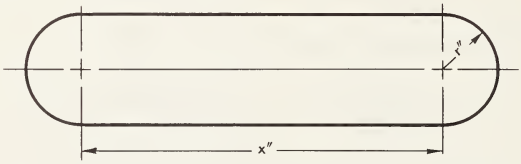
12. Copy and complete each of the following:

(*a*) 25 sq. m.	= ... sq. dm.	(*b*) 5 hectare	= ... sq. m.
(*c*) 23·5 sq. dm.	= ... sq. cm.	(*d*) 3·6 sq. cm.	= ... sq. m.
(*e*) 358 sq. mm.	= ... sq. cm.	(*f*) 628 sq. cm.	= ... sq. dm.
(*g*) 25 sq. dm.	= ... sq. m.	(*h*) 0·5 sq. m.	= ... sq. cm.

13. Calculate the per cent of material used when each of the shaded figures is cut from a rectangular piece of sheet metal.



14. A sheet-metal heating duct is made so that its cross-section has parallel sides with semicircular ends. Calculate the area of a duct the dimensions of which are indicated as follows:



(*a*) $x = 20$, $r = 6$; (*b*) $x = 15$, $r = 5$;
(*c*) $x = 18$, $r = 4\frac{1}{2}$; (*d*) $x = 16$, $r = 8$.

15. A printer received a supply of bond paper sheets 17″ by 22″. How many of these large sheets will be required to fill an order for 1,000 letter-heads $8\frac{1}{2}$″ by 11″?

16. How many sheets of paper $6\frac{1}{4}$″ by $9\frac{1}{2}$″ can be cut from a sheet of stock 17″ by 22″?

17. If a rectangle contains 4 square centimetres and is 50 millimetres long, what is its width?

18. In your school yard, mark out with suitable stakes an area equal to 1 acre.

19. If you buy a cottage lot with a frontage on the lake of 100 feet, how deep would it need to be to contain $\frac{1}{2}$ an acre?

20. Construct a parallelogram **ABCD** with $\angle \mathbf{B} = 45°$, $\mathbf{BC} = 3''$ and $\mathbf{AB} = 3''$. Construct and measure its altitude. Calculate the area of the parallelogram.

21. Two straight lines **AB** and **CD** bisect each other at right angles at **E**. If **AB** is 20″ long and **CD** is 14″ long and the points **A**, **C**, **B** and **D** are joined to form a quadrilateral, find the area of the quadrilateral.

22. In the triangle **ABC** the angle **B** is 90°. Calculate

(*a*) the length of the hypotenuse if **AB** is 21 inches and **BC** is 28 inches;
(*b*) the length of **AB** if **AC** is 78 cm. and **BC** is 30 cm.;
(*c*) the length of **BC** if **AC** is 17 inches and **AB** is 12 inches.

23. A softball diamond is a square 45 feet to a side. Calculate the distance from home to second base.

24. Find the square roots of the following numbers:

(*a*) 3,844; (*b*) 289; (*c*) 6,241;
(*d*) 841; (*e*) 676; (*f*) 7,396;
(*g*) 8.836; (*h*) 9,604; (*i*) 7,921.

25. Find the square root of each of the following:

(*a*) 364,816; (*b*) 257,049; (*c*) 197,136;
(*d*) 50,625; (*e*) 37,636; (*f*) 42,849.

26. Calculate correct to the nearest hundredth:

(*a*) $\sqrt{6}$, (*b*) $\sqrt{10}$, (*c*) $\sqrt{7}$.

C

27. A trapezoid has two parallel sides of 18″ and 24″ with the perpendicular distance between them 16″. Calculate its area.

28. A square **ABCD** 10″ to a side is formed into an octagon by cutting off the four corners with lines at 45° to the sides. If the distance from each corner of the square to the mark where the cut is to be made is 2·9″, find the area of the resulting octagon.

29. Compare by division the area of a circle 3″ in diameter with one 4″ in diameter. Express your comparison as a per cent.

30. If a piece of stock is 1 inch in diameter, find the side of the largest square which can be milled from this stock.

31. What is the diameter of a circle that is equal in area to the sum of the areas of two circles the diameters of which are 6″ and 8″ respectively?

32. A circular ventilating duct 15″ in diameter provides air for two branch ducts. If one of these has a diameter of 12″, what would be the largest diameter of the other circular duct which could also be accommodated in the system?

CHAPTER 8

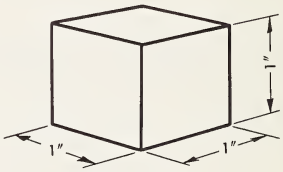
Measurement of Volume

All objects occupy space. The quantity of space occupied by an object is called its **volume.**

Units

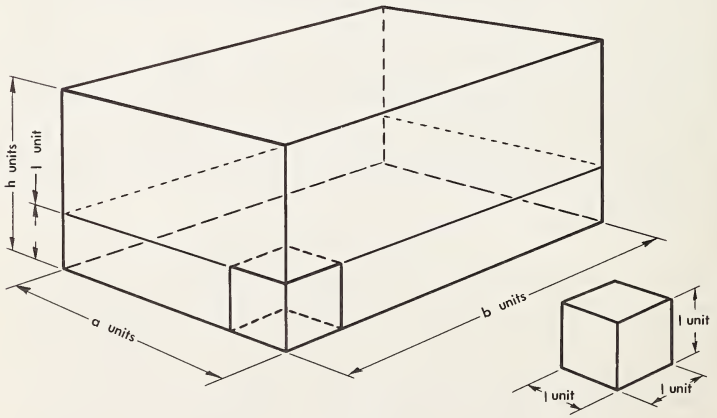
Before we can measure volume we must have a suitable unit of measurement. For area, the units of measurement are usually associated with the corresponding linear units. Likewise for volume, the cubic units are defined in terms of the corresponding linear units.

A cubic inch is the measurement of space occupied by a rectangular solid 1 inch long, 1 inch wide and 1 inch high. Other cubic units are defined in a similar manner.



The Volume of a Rectangular Solid

In general, the relationship between the volume of a rectangular solid and its linear dimensions may be established as follows:



Let the volume of the prism in cubic units be V.
Let the length of one side of the base in linear units be a.
Let the length of the other side of the base in linear units be b.
Let the height of the prism in similar linear units be h.
The number of cubic units in the first row of the base is b.
The number of these equal rows in the first layer is a.
Then the volume of the first layer in cubic units is $a \times b$.
The number of similar layers is h.
Therefore, the volume of the prism in cubic units is $ab \times h$.
But the volume of the prism in cubic units is V.

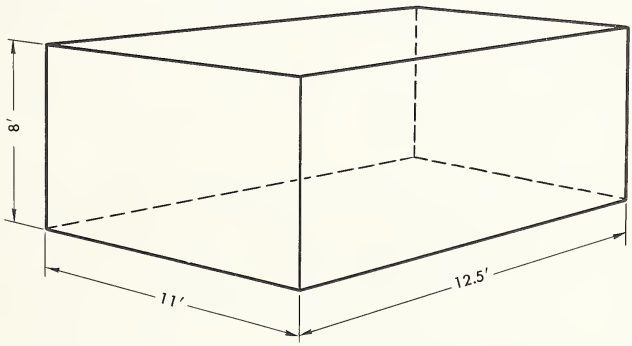
$$\therefore V = abh.$$

The measure of the volume of each layer, ab, is the same as the measure of the area of the base, A.
Then the volume of the prism in cubic units is expressed by Ah.

$$\therefore V = Ah.$$

This is the more general way of calculating a volume.

Example 1: Find the cubical contents of a room $12\frac{1}{2}$ feet long, 11 feet wide and 8 feet high.
Solution:



The relationship for the volume of the prism is $V = abh$.
The volume of the room in cubic feet is $12{\cdot}5 \times 11 \times 8$

$$= 1{,}100.$$

Example 2: A ton of coal occupies approximately 36 cubic feet. To what depth will 9 tons fill a bin 8 feet long and 6 feet wide?

Solution: The volume of the coal in cu. ft. is 9×36.
The area of the base of the bin in sq. ft. is 8×6.

The depth of the bin in feet is $\frac{9 \times 36}{8 \times 6} = \frac{27}{4} = 6\frac{3}{4}$.

The depth of the coal is 6′ 9″.

Exercise

A

1. Suggest suitable dimensions for four different rectangular solids each of which have a volume of 1 cubic inch.
2. State the volume of each of the following rectangular solids:
   (*a*) length 5′, width 4′, depth 2′;
   (*b*) length 8 cm., width 6 cm., height 2 cm.;
   (*c*) area of base 25 sq. in., height 8 in.;
   (*d*) area of base 40 sq. cm., height 9 cm.;
   (*e*) area of one end 16 sq. cm., length 8 cm.;
   (*f*) area of top 60 sq. ft., depth 9 ft.
3. What is the volume of each of the following cubes with side measurements of
   (*a*) 2 in., (*b*) 3 cm., (*c*) 4 ft., (*d*) 5 m., (*e*) 6 mm.?
4. If you doubled the edge of a cube, what change would you make in its volume?
5. If you tripled the edge of a cube, what change would you make in its volume?
6. A box is to be 10 in. long and 8 in. wide. How deep must it be to hold 480 cu. in?
7. A box is 3″ long and 6″ deep. How wide must it be to contain 36 cu. in.?
8. State orally the quotient when each of the following numbers is divided by 12: 264, 196, 3,624, 4,808, 607.

B

9. Find the cubical contents of a room 16 feet long, 11 feet 6 inches wide and 8 feet high.
10. A box of chalk is 6 inches long, $4\frac{1}{4}$ inches wide and $3\frac{3}{4}$ inches deep. What is the volume of the box?
11. Find the volume of the ice on a truck having a box 8′ long, 5′ 6″ wide, filled to a depth of $2\frac{1}{2}$ feet.

12. A farmer's grain bin 6′ 9″ long and 7′ 3″ wide is filled to a depth of 6′. What is its volume in cubic feet?

13. The ice on a rink is 2″ thick and covers a surface 35′ long and 24′ wide. Find its volume.

14. If a truck used to remove earth from an excavation carries 5 cubic yards at a time, how many trips will it make to remove the earth from an excavation 45 feet long, 24 feet wide and 9 feet deep?

15. Find the volume of a tool box if the area of its base is $2\frac{1}{4}$ square feet and its height is $1\frac{1}{2}$ feet.

16. The area of the end of a carton is 160 square inches. If it is 28 inches long, what volume will it contain?

17. A tea carton has an end area of $6\frac{1}{4}$ square inches. It is $6\frac{1}{2}$ inches long. What volume of tea does it contain?

18. A rectangular solid iron bar has a cross-section of $2\frac{1}{4}$″ by $\frac{3}{4}$″. How long must the bar be to contain 144 cubic inches?

19. For proper ventilation, a room should contain 3,264 cu. ft. of air space. The room is to be 24′ long and 16′ wide. How high should it be?

20. What is the volume in cubic feet of the metal in the base of a machine if the length is 27″, the width is 15″ and the thickness is 4″?

21. A rectangular cake tin is $7\frac{1}{2}$″ long, 6″ wide and 3″ deep. What volume of cake batter does it contain when it is half full?

22. A fast-freezing unit is 2′ 6″ long, 1′ 9″ wide and 9″ deep. Find the volume of this unit in cubic feet.

23. A gallon of liquid occupies 277·42 cubic inches. What is the height of a gallon measure with a square base 6″ to a side?

24. A rectangular fish tank is 20″ long and 12″ wide.

(*a*) What is the area of its base?

(*b*) To what depth will it have to be filled if it is to contain 1,920 cubic inches of water?

25. A square prism 8″ high has a volume of 200 cubic inches.

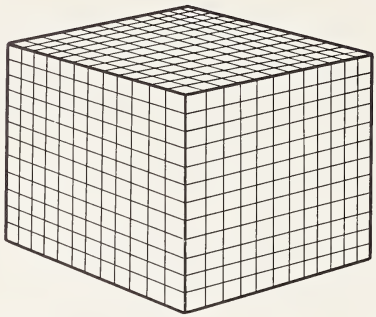
(*a*) What is the area of its base?

(*b*) What is the length of one side of its base?

26. The box of a gravel truck is 8 feet wide and 10 feet long. To what depth will it have to be filled in order to carry 4 cubic yards of gravel?

Relationships between English Cubic Measures

Examine a model of a cubic foot. A diagram of one is shown here.



1 cubic foot = 1,728 cubic inches.

Notice that there are 12 layers each of which is one inch thick. Each layer contains 12×12 cubic inches. Therefore,

$$1 \text{ cubic foot} = 12 \times 12 \times 12 \text{ cubic inches.}$$

Similarly

$$1 \text{ cubic yard} = 3 \times 3 \times 3 \text{ cubic feet.}$$

Cubic units larger than 1 cubic yard are rarely, if ever, used.

ENGLISH CUBIC MEASURE RELATIONSHIPS

1,728 cubic inches (cu. in.) = 1 cubic foot.

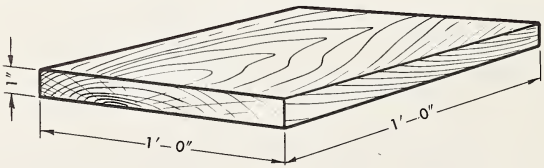
27 cubic feet (cu. ft.) = 1 cubic yard (cu. yd.).

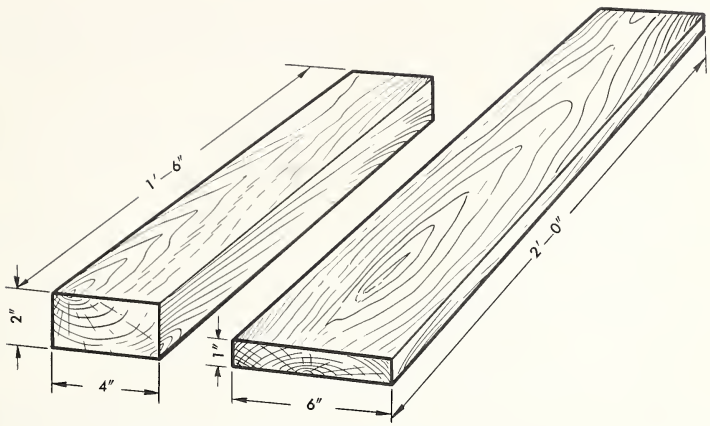
Special Units of Volume

For measuring quantities of wood a **cord** is used as the unit of measurement. A pile of wood 8 feet long, 4 feet wide and 4 feet high contains a standard cord.

1 cord of wood occupies 128 cubic feet.

For expressing the measure of quantities of lumber, a **board foot** is used as a unit of measurement. The standard board foot is a piece of lumber 1 foot long, 1 foot wide and 1 inch thick.





Each piece of lumber illustrated contains 1 board foot.

Example: A table top is to be made of planed lumber 2″ thick. The table top is 6 feet long and 30 inches wide. How many board feet of lumber are there in this table top?

Solution: The length of the table top in feet is 6.

The width of the table top in feet is $\frac{30}{12}$

$= 2{\cdot}5.$

The thickness of the table top in inches is 2.

Therefore, the quantity of lumber used in board feet

is $6 \times 2{\cdot}5 \times 2$

$= 30.$

Exercise

A

1. A cube measures 4″ by 4″ by 4″. How many cubic inches does it contain? Express its volume in cubic feet.

2. Find the volume of a pillar with a base of 1 sq. ft. and a height of 26 ft.

3. Suggest at least four sets of dimensions for a soap package if its volume is to be 120 cubic inches.

4. (*a*) Change $\frac{1}{2}$ cu. ft. to cu. in. (*b*) Change 2 cu. yd. to cu. ft.
(*c*) Express 9 cu. ft. as cu. yd. (*d*) Express 144 cu. in. as cu. ft.
(*e*) How many board feet are in a piece of shelving 16′ long, 12″ wide and 1″ thick?
(*f*) How many board feet are in a plank 6′ long, 12″ wide and 2″ thick?

5. If a roof is to be sheeted with 1″ lumber and has an area of 560 sq. ft., how many board feet of lumber will be required?

6. Mr. Brown needs sufficient 1″ lumber to line 1,200 square feet of his cottage. If 10% extra is allowed for matching and waste, how many board feet of lumber will he require?
7. How many board feet of lumber are in a 2″ by 4″ scantling if it is 12 feet long?
8. A rectangular prism has an end section 5″ by 6″ and is 20″ high. What would be its volume?
9. Find the volume in cubic feet of a rectangular prism that has a base 12″ by 6″ and an altitude of 48″.
10. Find the number of board feet required for the floor of a room 40 feet by 30 feet if the lumber is $1\frac{1}{2}$″ thick.

B

11. Make a drawing of a cubic yard showing the front face drawn to scale. By means of cross-lines show that there are 27 cubic feet in a cubic yard.
12. The box of a gravel truck is 9 feet long, 6 feet wide and 27 in. deep. When it is full, how many cubic yards of gravel does it contain?
13. A man wishes to top-dress his lawn to a depth of $1\frac{1}{2}$″ with loam at $4 per cubic yard. What will be the cost of the loam if it is to cover a rectangular area 35 ft. long and 18 ft. wide?
14. If the inside dimensions of a box are 3′ 3″, 2′ 6″ and 12′ 0″, find its capacity in cubic feet.
15. A box with a lid is made of wood 1″ thick. If the outside dimensions are 3′, 6′ 8″ and 16″, find its inside volume in cubic feet.
16. What is the greatest number of cereal packages the dimensions of which are 4″ by 2″ by 6″ that can be fitted into a carton 24″ long, 18″ wide and 15″ deep?
17. What quantity of lumber is contained in 1 piece of 2″ by 4″ scantling which is 16 feet long?
18. A man uses 1″ sheeting to cover the two sides of a gable roof which is 24 feet long and 14 feet along the rafters. What is the cost of the sheeting at $125 per 1,000 board feet?
19. The studding for a cottage required 44 pieces of 2″ by 4″ each 8 feet long. Find the cost of this material at $200 per thousand board feet.
20. A lumber yard employee finds that a pile of lumber is 10 feet long, 8 feet wide and 6 feet high. How many board feet of lumber are in this pile?
21. A log is 12 feet long and it is estimated that, after the rounded edges are removed, the end would be 14″ by 16″. How many board feet of lumber should it produce?

22. Find the cost of flooring the attic of your home with inch lumber if the floor is 28 feet long and 18 feet wide. Lumber costs $210 per thousand board feet.

23. Find the cost of 15 scantlings each 2″ by 4″ by 12 feet long and 24 planks each 2″ by 6″ by 14 feet long if lumber costs $190 per thousand.

24. How many cords of wood are there in a pile 24 feet long if the length of each stick is 4 feet and the height of the pile is 4 feet?

25. Find the cost of a pile of wood 15 feet long, 4 feet high if each stick is 2 feet long. A cord of wood costs $25.

26. If wood costs $24 per cord and each stick is 12 inches long, find the cost of 3 piles that are 3 feet high and 18 feet long.

27. A rectangular tank is 16 inches long and 8 inches wide. A cubic foot of sand is poured into it and levelled off. How deep will the sand be?

Weight and Capacity

In the English system there is an almost unlimited number of different units of measurement for weights and capacities. Some of these are used for highly specialized measurements. Troy units of weight are used for expressing the weight of precious metals, stones, etc. Apothecaries' units are, for the most part, confined to measurements of drugs. In our work we shall confine our attention to more generally used units, the Avoirdupois units for weight and the Imperial and the United States units for capacity.

ENGLISH UNITS FOR LIQUID MEASURE

The relationships among the units of capacity are the same in both the Imperial and United States systems, but the units themselves are not the same.

2 cups = 1 pint.
2 pints = 1 quart.
4 quarts = 1 gallon.

Imperial Units	*United States Units*
1 gallon = 160 fluid ounces.	1 gallon = 128 fluid ounces.
* 1 gallon occupies 277·420 cu. in.	(133·25 Canadian fluid ounces)
1 cu. ft. contains 6·232 gallons.	1 gallon occupies 231 cu. in.
1 gallon of water weighs 10 lb.	1 cu. ft. contains 7·48 gallons.

5 Imperial gallons are the approximate equivalent of 6 United States gallons.

ENGLISH UNITS FOR DRY MEASURE

2 pints = 1 quart.
4 quarts = 1 gallon.
2 gallons = 1 peck.
4 pecks = 1 bushel.

* Standards Division, Government of Canada, Toronto office 1954.

The Imperial units used in liquid measurement have the same capacity as those in dry measurement. This is not the case in the United States units.

1 Imperial bushel = 2219·36 cu. in.

1 United States bushel = 2150·42 cu. in.

English Units of Weight

The original basic unit of weight in all English systems is the grain—the weight of a plump grain taken from the middle of a head of wheat. It served as a standard unit of weight in medieval England. One pound Avoirdupois is 7,000 grains. One pound, in Canada, is defined by statute to be $\frac{45,359,243}{100,000,000}$ of an International Kilogram.

16 drams	= 1 ounce.
16 ounces (oz.)	= 1 pound (lb.).
100 pounds	= 1 hundredweight (cwt.).
20 hundredweight	= 1 ton.
1 short ton	= 2,000 pounds.

Still common in England are

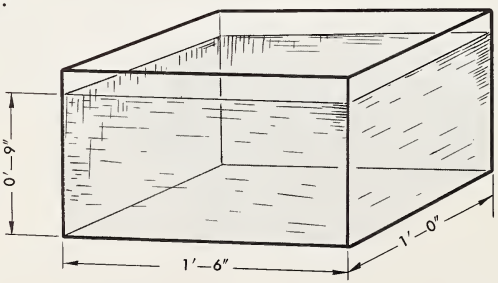
14 pounds	= 1 stone.
8 stones	= 1 hundredweight (112 lb.).
20 hundredweight	= 1 ton.
1 long ton	= 2,240 pounds.

Volume, Weight and Capacity

Since cubic units are related to certain corresponding linear units of an object, they often serve as a connecting link in the calculations of capacity or weight when only linear measurements of the object can be found.

Example: What is the capacity in gallons of a rectangular aquarium 1′ 6″ long, 1′ 0″ wide when filled with water to a depth of 0′ 9″? What will be the weight of the water contained in it?

Solution:



The volume of the tank in cu. ft. is $1{\cdot}5 \times 1{\cdot}0 \times 0{\cdot}75 = 1{\cdot}125$
Since 1 cubic foot holds 6·23 gallons, the capacity of the tank in gallons is $1{\cdot}125 \times 6{\cdot}23 = 7{\cdot}008$ or 7 approximately.
Since 1 gallon of water weighs 10 lb., the weight of water in the tank in pounds is $7{\cdot}008 \times 10 = 70$ approximately.

Exercise

A

1. Complete the following statements:

(*a*) 8 cups = ... pints. (*b*) 6 pints = ... quarts.
(*c*) 12 quarts = ... gallons. (*d*) 5 gallons = ... pints.
(*e*) 2 lb. = ... oz. (*f*) 5 lb. 8 oz. = ... lb.
(*g*) 4 bushels = ... pecks. (*h*) 20 gallons = ... bushels.
(*i*) 30 Imp. gal. = ... U.S. gal. (*j*) 100 Imp. gal. = ... U.S. gal.
(*k*) 12 U.S. gal. = ... Imp. gal. (*l*) 60 U.S. gal. = ... Imp. gal.
(*m*) 25 gal. water weigh ... lb. (*n*) 45 gal. water weigh ... lb.
(*o*) 1 cu. ft. contains ... gal. (*p*) 10 cu. ft. contain ... gal.

2. How many pounds heavier is a long ton than a short ton?

3. If a boy weighs 7 stone 5 lb., what is his weight in lb.?

4. If your mother weighs 8 stone, what is her weight in lb.?

5. How many cups of milk are there in $\frac{1}{2}$ gallon?

6. If milk has approximately the same weight per unit volume as water, how many pounds will 8 gallons of milk weigh?

7. How many ounces are there in

(*a*) 5 pounds; (*b*) 64 drams; (*c*) 10 pounds; (*d*) 1 hundredweight?

B

8. A classroom is 45′ long, 35′ wide and 15′ high. If the air in the classroom is to be changed 5 times every hour, what volume of air must enter the room each minute?

9. A tank for quenching steel is 24″ long, 18″ wide and 30″ deep. How many gallons of oil will be required to fill it within 6″ of the top? Find the price of the oil at 45 cents per gallon.

10. If a hardware merchant purchases a 5-gallon tin of turpentine, how many pint bottles should he be able to fill?

11. If a grain bin 10 feet long and 5 feet wide is filled to a depth of 4 feet, how many Imperial bushels of grain does it contain?

12. Copy and complete the following statements:

(*a*) 10 gallons = ... pints. (*b*) 100 quarts = ... gallons.
(*c*) 1 gallon = ... cups. (*d*) 10 cups = ... quarts.

(*e*) 8 Imperial gallons occupy ... cu. in.
(*f*) 8 United States gallons occupy ... cu. in.
(*g*) 1,386 cu. in. contain ... Imperial gallons (approx.).
(*h*) 1,386 cu. in. contain ... United States gallons.
(*i*) 10 cu. ft. contain ... Imperial gallons.
(*j*) 10 cu. ft. contain ... United States gallons.

13. A bushel of wheat weighs 60 pounds. Estimate the number of kernels contained in it.

14. What is the weight of a shipment of 50 bars of 1 in. by 2 in. flat cold rolled steel if each bar is 6 ft. long? 1 cu. in. of steel weighs 0·2834 lb.

15. A rectangular cistern 8 ft. long and 6 ft. wide is filled to a depth of 5 ft. How many gallons of water does it contain?

16. If 1 foot of water is drawn off a rectangular cistern which is 10 ft. long and 8 feet wide, how many gallons will have been removed?

17. A flat roof is cooled in summer by flooding with water to a depth of 2″. If the roof is 32 feet long and 24 ft. wide, how many gallons of water are required?

18. During a $\frac{1}{2}$″ rainfall, how many gallons of water fell on a paved street 33′ wide and 600′ long?

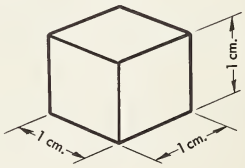
19. Which of the following relationships would be more nearly accurate in estimating the capacity of a grain bin:

(*a*) 3 bushels occupy 4 cu. ft. or (*b*) 4 bushels occupy 5 cu. ft.?

Metric Units of Volume, Capacity and Weight

In the Metric system the most commonly used units of volume are the cubic centimetre, the cubic decimetre and the cubic metre.

The standard cubic centimetre is a cube 1 centimetre in length, 1 centimetre in width and 1 centimetre in height.



1 cubic centimetre

The cubic decimetre is the volume used as the basic metric unit of capacity. As a unit of capacity it is called a litre and is used for measuring liquids and gases. A litre is almost equivalent to a United States quart.

Metric Units of Volume

1,000 cubic centimetres (c.c.) = 1 cubic decimetre.
1,000 cubic decimetres (c.dm.) = 1 cubic metre (c.m.).

When it is necessary to express in metric units the volume of a solid, the dimensions of which are given in English units, it is advisable first to change

the English linear measurements to their equivalent Metric linear measurements. The change could be made, however, by first calculating suitable equivalent measures. For example, 1 inch = 2·54 cm. Therefore, the measure of one cubic inch in cubic centimetres is $2{\cdot}54 \times 2{\cdot}54 \times 2{\cdot}54$

= 16·387064 exactly

or 16·4 approximately.

Metric Units of Capacity

The other units of capacity in the Metric system are derived from the litre in the same way as the other units of length are derived from the metre.

1 kilolitre (Kl.) = 1,000 litres.

1 hectolitre (Hl.) = 100 litres.

1 dekalitre (Dl.) = 10 litres.

The basic unit for capacity is 1 litre.

1 decilitre (dl.) = $\frac{1}{10}$ litre.

1 centilitre (cl.) = $\frac{1}{100}$ litre.

1 millilitre (ml.) = $\frac{1}{1000}$ litre.

Relationships Between Units of Capacity and Volume

By definition, 1 litre = 1 cubic decimetre.

Therefore, 1 litre = 1,000 cubic centimetres.

1 millilitre = 1 cubic centimetre.

Metric measurements of capacity may be converted to Imperial measurements of capacity or vice versa by applying the relationships:

1 litre = 0·22 gallon.

1 gallon = 4·546 litres.

Metric Units of Weight

The basic unit of weight is the gram. *A gram is the weight of* 1 *cubic centimetre of water*. The other units of weight are derived from the gram in the usual metric manner.

1 kilogram (Kg.) = 1,000 grams.

1 hectogram (Hg.) = 100 grams.

1 dekagram (Dg.) = 10 grams.

The basic unit of weight is 1 gram.

1 decigram (dg.) = $\frac{1}{10}$ gram.

1 centigram (cg.) = $\frac{1}{100}$ gram.

1 milligram (mg.) = $\frac{1}{1000}$ gram (g.).

To express the weight of large masses the metric ton is used as a unit of weight.

1 metric ton = 1,000 kilograms.

For converting measurements of weight from the English to the Metric systems and vice versa the following relationships may be used:

1 pound = 453·6 grams.
1 kilogram = 2·2046 pounds.

Exercise

A

1. Complete the following statements:

(*a*) 1 litre = ... millilitres.
(*b*) 3 litres = ... millilitres.
(*c*) 0·6 litre = ... millilitres.
(*d*) 500 millilitres = ... litres.
(*e*) 2 c.dm. = ... litres.
(*f*) 56 c.dm. = ... litres.
(*g*) 1 litre = ... c.c.
(*h*) 10 litres = ... c.c.
(*i*) 345 c.c. = ... litres.
(*j*) 345 millilitres = ... litres.
(*k*) 345 c.c. = ... ml.
(*l*) 35 millilitres = ... c.c.

2. Complete the following statements:

(*a*) 1 c.c. of water weighs ... grams.
(*b*) 1000 c.c. of water weigh ... grams.
(*c*) 1 litre of water weighs ... grams.
(*d*) 1 litre of water weighs ... kilograms.
(*e*) 25 millilitres of water weigh ... grams.

3. Express as grams

(*a*) 5 kilograms 6 grams; (*b*) 20 grams 3 milligrams;
(*c*) 1 kilogram 54 grams 25 milligrams.

B

4. A box measures 30·4 cm. long, 15·2 cm. wide and 15·2 cm. high. Find its volume in cubic centimetres. What is its capacity in litres?

5. A piece of iron bar used in a chemistry experiment is 13·2 cm. long, 5·2 cm. wide and 0·5 cm. thick. Find its volume in cubic centimetres. Find its weight in grams if 1 c.c. of iron weighs 7·7 grams.

6. If 1 c.c. of lead weighs 11·38 g., what would 1 c.m. of lead weigh?

7. A rectangular-shaped tank is 2·35 m. long, 65 cm. wide and 24 cm. deep.

(*a*) What is its volume in cubic metres?
(*b*) What is its capacity in kilolitres?
(*c*) What is its weight when filled with water?

8. The outside dimensions of an open tank are 2·78 m. long, 78 cm. wide and 59 cm. deep. If the tank is made of concrete 15 cm. thick, what is its inside volume in cubic metres?

9. Copy and complete the following:

(*a*) 256 c.dm. = ... c.m. (*b*) 527 c.c. = ... litre.
(*c*) 3·75 litres = ... c.c. (*d*) 100 litres = ... gallons.

(*e*) 10 gallons = . . . litres. (*f*) 10 lb. = . . . Kg.
(*g*) 5 Kg. = . . . lb. (*h*) 1 oz. = . . . g.

10. (*a*) What is the weight in pounds of a metric ton? (*b*) Find the difference in weight between (*i*) a metric ton and a long ton; (*ii*) a metric ton and a short ton.

C

11. Find the length of the diagonal of a cube each side of which is
(*a*) 10″, (*b*) 3″, (*c*) 8″.

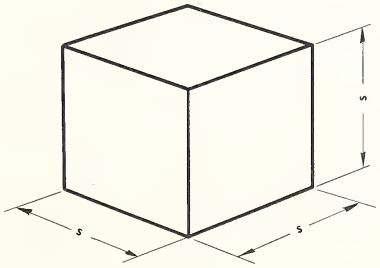
Cubes and Cube Root

A rectangular solid which has the same length, width and thickness is called a **cube.** We can see cubes in ice from the refrigerator, sugar cubes for tea and coffee, etc. When a solid is described as a cube, only one linear dimension is necessary to determine the other measures.

Example: Find the volume of a cube which is 10 inches long.

Solution: Volume of cube in cubic inches is $10 \times 10 \times 10$
$= 1000.$

A General Cube



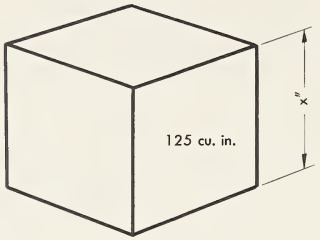
Let the volume of the cube in cubic units be V.
Let the length of the cube in linear units be s.
The relationship connecting V and s would be

$$V = s \times s \times s$$
$$V = s^3$$

Cube Root

Sometimes it is necessary to find the *size* of a cube which has a special volume. To do so we would have to reverse the process of finding the volume.

Suppose that a certain cube must contain 125 cubic inches. What would be the length of each edge?



The first relationship to recognize is that every cube has three equal dimensions. Our problem resolves itself into finding one of the three equal factors of 125.

Since $125 = 5 \times 5 \times 5$, the length of one edge in inches is 5.

The number 5 is called the **cube root** of 125.

The symbol to indicate cube root is $\sqrt[3]{\ }$.

Since $125 = 5 \times 5 \times 5$, therefore $\sqrt[3]{125} = 5$.

Similarly $\sqrt[3]{1000} = 10$ because $10 \times 10 \times 10 = 1000$.

Example: Find the cube root of 3375.

Solution: $3375 = 5 \times 5 \times 5 \times 3 \times 3 \times 3$.

Arrange the factors into three equal groups

$= 5 \times 3 \times 5 \times 3 \times 5 \times 3$

$= 15 \times 15 \times 15$.

Therefore, $\sqrt[3]{3375} = 15$.

5	3375
5	675
5	135
3	27
3	9
	3

Exercise

A

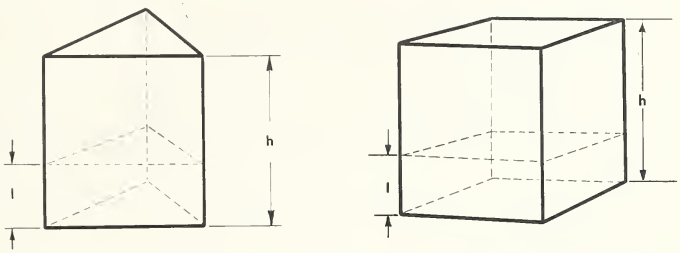
1. Find the value of 4^3, 5^3, 7^3, 1^3, $(0{\cdot}1)^3$, $(0{\cdot}3)^3$, $(\frac{1}{2})^3$.
2. A cube of sugar is $\frac{1}{2}''$ to a side. How many cubes of sugar could be made from 1 cu. in. of sugar?
3. Find the cube root of 8, 27, 64, 216 and 1,728.
4. If a certain cube contains 343 c.c., what is the length of one side?

B

5. Find the volume of a cube if its length is 3·2 cm.
6. If there is no waste, how many cubes of sugar $\frac{1}{2}''$ to a side can be made from 1 cubic foot of sugar?
7. An ice tray is 12″ long, 3″ wide and $1\frac{1}{2}''$ deep. How many cubes of ice, $1\frac{1}{2}''$ to a side, does it contain if no allowance is made for spacers?

8. Find by factoring the cube root of:
   (*a*) 8,000; (*b*) 5,832; (*c*) 9,261; (*d*) 15,625; (*e*) 13,824.
9. If a cube has a volume of 4,096 c.c., find its length in centimetres.
10. Find the length of a cube the volume of which is the same as that of a rectangular solid 81 feet long, 45 feet wide and 25 feet deep.

Volume of a Prism



Cut off a layer 1 unit in depth. The measure of the volume of this layer in cubic units is the same as the measure of the area of the base in square units.

Let the volume of the prism in cubic units be V.

Let the area of the base of the prism in square units be A.

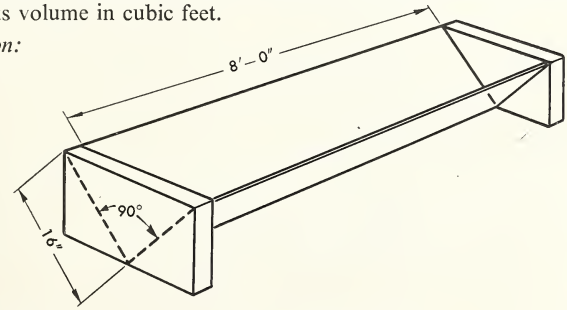
Let the height of the prism in corresponding linear units be h.

Then the volume of the prism in cubic units is $A \times h$.

$$\therefore V = Ah.$$

Example: A trough that is 8 feet long has a cross-section which is an isosceles right-angled triangle with the sides about the right angle 16 inches in length. Find its volume in cubic feet.

Solution:



Here the end of the trough is the base of the prism.

The area of the triangular base in sq. ft. is $\frac{1}{2} \times \frac{16}{12} \times \frac{16}{12}$.

The volume of the prism in cu. ft. is $\frac{1}{2} \times \frac{16}{12} \times \frac{16}{12} \times 8$.

The volume of the trough in cu. ft. $= 7{\cdot}1$.

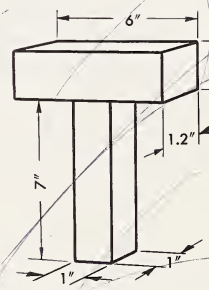
Exercise

A

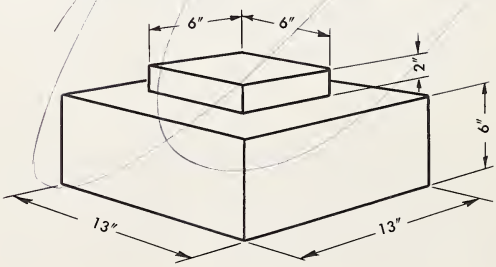
1. An hexagonal prism 6 ft. long has an area of the base of 1 sq. ft. What is its volume?
2. A triangular prism 20″ long has an area of base of 84 sq. in. What is its volume?
3. The volume of a prism is 300 cu. in. If the prism is 15 inches long what is the area of its base?
4. A square prism stands on a base 2″ to a side and has a volume of 64 cu. in. What is its length?
5. A rectangular bar of steel is 2″ by $\frac{1}{4}$″ and is 12 feet long. What is its volume?
6. The volume of a rectangular rod is 1 cu. in. If it is 6″ long, what is the area of its end?

B

7. How many tons of coal can be put in a bin 8′ 6″ long, 8′ wide and 6′ high if one ton occupies 36 cu. ft.?
8. Find the volume of this T bar.

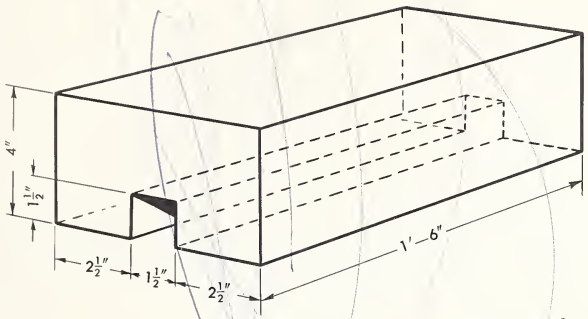


9. Find the total volume of this block.

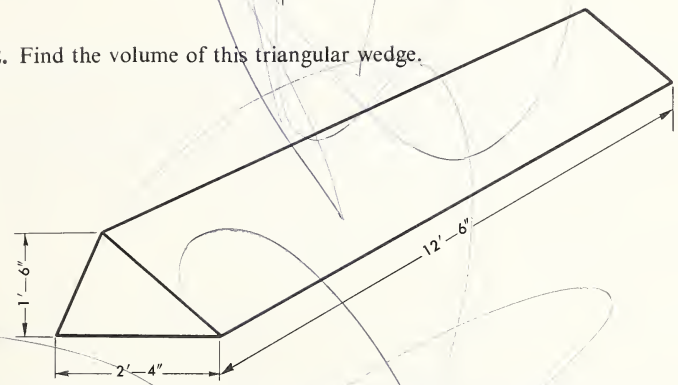


10. Find the weight of a mild steel bar 12′ long, $2\frac{1}{2}''$ wide and $\frac{1}{4}''$ thick if mild steel weighs 0·28 lb. per cu. in.

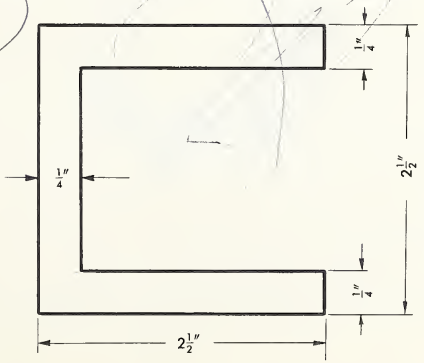
11. Find the volume of this block.



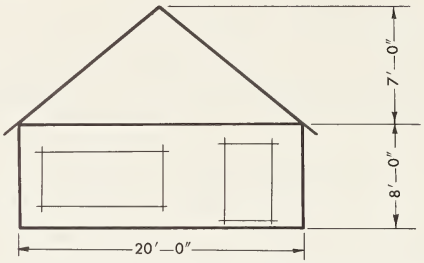
12. Find the volume of this triangular wedge.



13. Find the volume of 14′ of channel iron with cross-section as shown.



14. Find the cubical contents of a cottage the end of which is shown with its dimensions. The cottage is 24′ long.



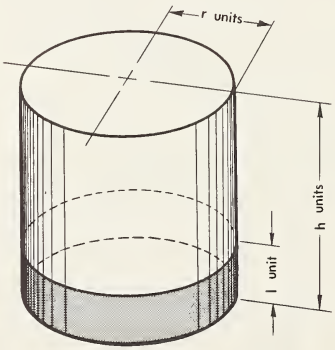
15. A rectangular tank with a length of 2′ and width of 1′ is partly filled with water. When a solid piece of cast iron is dropped into the tank, the water rises 3″. Find the volume of the iron block.

C

16. (*a*) The first of two sheet metal water tanks is 9′ long, 6′ wide and 4′ deep; the second is 10′ long, 8′ wide and 2′ deep. Compare by subtraction the volumes of the two tanks.

(*b*) If each of the tanks mentioned above has a lid, show that the tank with the larger volume takes the lesser amount of sheet metal to make. Explain this.

Volume of a Cylinder



Let the measure of the volume of the cylinder in cubic units be V.

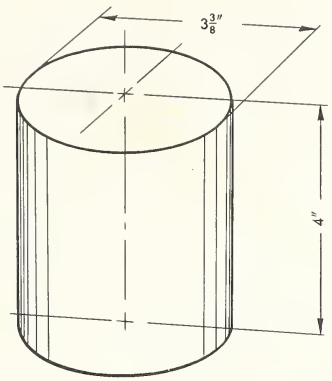
The area of the base in square units is πr^2.

Now using the relationship for a prism $V = Ah$,
we find the volume of the cylinder in cubic units is $\pi r^2 h$.

$$\therefore V = \pi r^2 h$$

Example: A can of peas is $3\frac{3}{8}''$ in diameter and $4''$ in height. Find its volume in cubic inches.

Solution:



The area of the base in sq. in. is $3{\cdot}14 \times \frac{27}{16} \times \frac{27}{16}$.

The relationship for the volume of a cylinder is $V = \pi r^2 h$.

The volume of cylinder in cu. in. is $3{\cdot}14 \times \frac{27}{16} \times \frac{27}{16} \times 4$

$= 35{\cdot}76$.

The volume of a can of peas in cu. in. $= 35{\cdot}8$ approx.

Exercise

A

1. The area of a circle at the base of a cylinder is 24 sq. in. If the cylinder is 5 in. high, what is its volume?
2. A rectangular brick with a volume of 64 cu. in. is dropped into a cylinder containing water. If it raises the height of the water by 2", what is the area of the base of the cylinder?
3. A rectangular piece of iron 8" by 5" by 2" is dropped into a cylinder containing water. If it raises the water $\frac{1}{2}''$, what is the area of the base of the cylinder?
4. The value of π correct to 8 places of decimals is 3·14159625. What is the value of π correct to (*a*) 7 places, (*b*) 6 places, (*c*) 4 places.

B

5. The diameter of the base of a cylindrical column is 2′ 4″ and the perpendicular height is 27′. Find its volume. Use $\pi = 3\frac{1}{7}$.
6. A cylinder has a diameter of 1′ 9″ and a height of 6′. Find its volume in cubic feet. Use $\pi = 3\frac{1}{7}$.

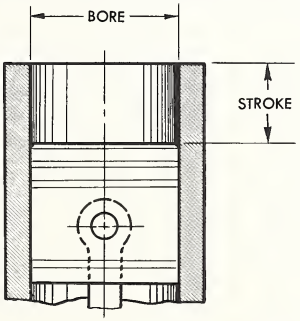
7. A farmer's water tank, cylindrical in form, is 10′ in diameter and 4′ deep. How many gallons will it hold if 1 cu. ft. contains approximately $6\frac{1}{4}$ gallons? Use $\pi = 3{\cdot}14$.
8. A circular iron man-hole cover is 27″ in diameter and 1″ thick. If iron weighs 4·8 oz. per cu. in. find its weight in pounds.
9. An underground gasoline tank is 5′ in diameter and 15′ long. Find its capacity in gallons.
10. How many cubic feet of earth had to be removed from a cylindrical well 4′ in diameter and 16′ deep?
11. Find the number of cubic decimetres in a tank 14 cm. in diameter and 40 cm. in length. What is the capacity of this tank in litres?
12. A measuring can is $18\frac{1}{2}$″ in diameter and 8″ deep. What is its volume?
13. A gallon oil can 10″ in diameter is partially filled with water. A stone is dropped into the water and the water rises $3\frac{1}{2}$″. Find the volume of the stone.
14. A circular vat which is 12 feet in diameter is filled with paint to a depth of 6 feet. If the depth of paint is lowered by 18″, how many gallons of paint have been drawn off? 1 cu. ft. $= 6\frac{1}{4}$ gallons.
15. What is the weight of a column of water which fills a 3″ diameter down pipe to a height of 24 feet? 1 cu. ft. of water weighs $62\frac{1}{2}$ lb.
16. A water main has an inside diameter of 2″. Water flows through it at the rate of 1 foot per second.
    (*a*) How many cu. in. of water would flow out in 1 minute?
    (*b*) How many gallons of water would flow out in 1 hour?
    Use 277 cu. in. $=$ 1 gallon.
17. A water pipe is $\frac{1}{2}$″ in diameter. Water is flowing through this pipe at the rate of 2 feet per second. How many cubic inches of water will flow out in 1 second? How many gallons will flow out per hour?

Review Exercise

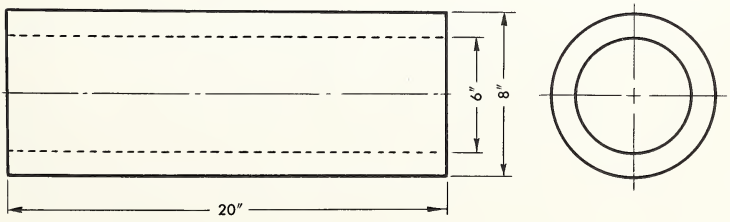
B

1. (*a*) What is the volume of a rectangular steel bar $4\frac{1}{2}$″ long if the cross-sectional dimensions are $\frac{3}{4}$″ by $1\frac{3}{4}$″?
   (*b*) What is the weight of this bar if steel weighs 0·28 lb. per cu. in.?
2. A coal bin is 8 ft. deep. The floor measurements are 12 ft. by 9 ft. If a ton of coal occupies approximately 36 cu. ft., find (*a*) the amount of coal in tons which should fill it to within 2 ft. of the top; (*b*) the cost of the coal at $25 per ton.
3. An Imperial gallon contains 277·4 cu. in. What would be the volume in cubic inches of 1 quart; 1 pint?

4. A U.S. gallon contains 231 cu. in. It is often said that 6 U.S. gallons are equivalent to 5 Canadian gallons. What is the actual difference in these two volumes?
5. A cubical box open at the top requires 245 square feet of material to line the sides and bottom.
   (*a*) What are the inside dimensions of the box?
   (*b*) How many gallons will it hold if 1 cu. ft. contains 6·23 gallons?
6. Find the volume of a triangular prism 9″ high. The base of the triangle is 5″ and the altitude to this base is 4″.
7. How many gallons of fuel-oil are in a cylindrical tank $5\frac{1}{4}$ feet in diameter if it is filled to a depth of 8 ft.?
8. The school has ordered 24 round cast-iron bars each $\frac{3}{4}$″ in diameter and 12 feet long. Find:
   (*a*) the volume of 1 bar;
   (*b*) the total weight of the bars if 1 cu. in. of cast iron weighs 0·26 lb.
9. Find the displacement of a piston in a cylinder
   (*a*) if the bore is $3\frac{1}{2}$ inches and the stroke 4 inches;
   (*b*) if the bore is $2\frac{3}{4}$ inches and the stroke is $3\frac{1}{2}$ inches.



10. How many cubic feet of concrete will be needed to make 2,000 concrete drain tiles as shown in the diagram?



11. If steel sells for \$150 per ton, estimate the cost of the following shipment. Assume 1 cu. in. weighs 0·28 lb.

6 bars of S.A.E. 1020, $\frac{7}{8}$″ by $\frac{7}{8}$″ by 12′ long;
4 bars of S.A.E. 1040, $\frac{5}{8}$″ by $1\frac{1}{2}$″ by 12′ long;
3 bars of S.A.E. 1124, $\frac{1}{4}$″ by $1\frac{1}{2}$″ by 12′ long.

All bars are rectangular in cross-section.

12. If the water on a flat roof 20 feet by 30 feet is maintained at a depth of 2 inches, what is the weight of the water?

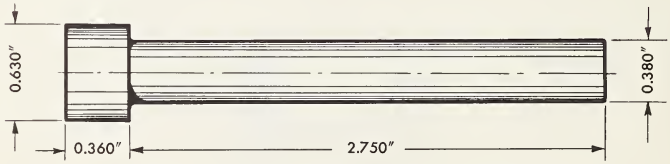
C

13. How many gallons of water will fall on an acre of ground during a $\frac{1}{2}$-inch rainfall?

14. If a lawn is 30 feet by 50 feet, how many gallons of water would have to be applied by a hose to supply as much as a $\frac{1}{4}$-inch rainfall?

15. If a water supply tank is 20 feet in diameter, by how much will the level of the water be lowered when 1,000 gallons are withdrawn?

16. The dimensions for a blank standard $\frac{3}{8}$″ fillister-head screw, when in the rough, are shown on the diagram.



(*a*) Find its volume.
(*b*) Find the weight of 1,000 blank screws if 1 cu. in. weighs 0·28 lb.

CHAPTER 9

General Number

Indicated Operations

We have used letters to represent measures of length, area and volume and to express familiar relationships between these measures. For example, if the lengths of the sides of a rectangle in linear units are x and y and the area in corresponding square units is z, then the relationship between x, y, and z is expressed by the statement $z = yx$.

More generally, since x and y are any two numbers, then the four fundamental operations with these numbers are indicated as follows:

(a) the sum of x and y is represented by $x + y$ or $y + x$;
(b) the difference when x is subtracted from y is $y - x$;
(c) the product of x and y is xy or yx;
(d) the quotient when x is divided by y is $\frac{x}{y}$.

Changing the Form of a Measure

Under certain circumstances the same measure may be expressed in different forms. The following are equivalent forms of the same measures:

(a) $x + x + x = 3x$; (b) $x \times x \times x = x^3$;
(c) $2(x + y) = 2x + 2y$; (d) $3x + 4x = 7x$.

Example 1: I am k years old.

(a) If Jim is twice as old as I am, how old is Jim?
(b) How old will I be next year?
(c) How old was I five years ago?

Solution: (a) Jim's age in years is $2 \times k = 2k$.
(b) Next year I will be $k + 1$ years old.
(c) Five years ago I was $k - 5$ years old.

Example 2: Simplify the following measures:

(a) $aaa + aa + aa$; (b) $9x - 5x$.

Solution: (a) $aaa + aa + aa$
$= a^3 + a^2 + a^2$
$= a^3 + 2a^2$.

(b) $9x - 5x$
$= 4x$.

Exercise

A

1. Express in the simplest form each of the following:

(*a*) $a + a$,	(*b*) $a \times a$,	(*c*) $a + a + b + b$,
(*d*) aaa,	(*e*) $aaaa$,	(*f*) $a \times b \times c$,
(*g*) $aa + aa$,	(*h*) $3a + 5a$,	(*i*) $2a + b$,
(*j*) $x + x + x + x$,	(*k*) $x \times x \times x \times x$,	(*l*) $3 \times a \times a$,
(*m*) $2 \times y \times y \times y$,	(*n*) $x + xx$,	(*o*) $y + yy + yyy$.

2. Use the signs of mathematics to indicate:

(*a*) the sum of the three numbers a, b and c;
(*b*) the difference between m and n when n is subtracted from m;
(*c*) the product of the three numbers a, b and c;
(*d*) the product of a and b divided by c.

3. I have x cents in one pocket and y cents in another. How much money have I?

4. If I am k years old and you are twice as old, what is your age?

5. What are two possible meanings for $3x$?

6. A boy is n years old now. How old will he be in 10 years? How old was he 5 years ago? In how many years will he be $2n$ years old?

7. A rectangle is c inches long and b inches wide. What is its perimeter?

8. What is the cost of 8 oranges at 5 cents each? What is the cost of x oranges at 5 cents each? What is the cost of x oranges at y cents each?

9. Complete the following sentences:

(*a*) The sum of x and y is . . . . .
(*b*) The product of a and c is . . . . .
(*c*) The quotient of n by m is . . . . .
(*d*) The difference of r and p is . . . . or . . . . .
(*e*) Twice the product of c and d is . . . . .
(*f*) The square of y is . . . . .
(*g*) The cube of x is . . . . .

10. What is the value in cents of 7 dimes; of n dimes?

11. How many minutes are in 3 hours; in t hours; in m seconds?

12. How many articles are there in 3 dozen; in n dozen?

13. What is the next integer above 23; above x; below x?

14. A rocket has a speed of 1,200 m.p.h. How far will it travel in 4 hours; in t hours; in n minutes?

15. If a boy can cut 600 sq. ft. of lawn in 1 hour, how much should he be able to cut in 10 minutes; in x minutes; in y hours?

16. State the results of the following operations:

(a) 2 is added to n; (b) x is multiplied by 3;
(c) three less than n; (d) 14 is divided by t;
(e) increase k by 7; (f) take half of z and add 2;
(g) half the sum of k and 2; (h) decrease m by 7;
(i) the difference of x and y divided by 5;
(j) double the product of a, b and c;
(k) the quotient when the product of m and n is divided by k;
(l) the product of 2 and k divided by 3;
(m) two thirds of k.

B

17. Write in shorter form.

(a) $x + x + x + x$, (b) $3a + 3a + 4a$,
(c) $4b + b$, (d) $bb + bb + bb$,
(e) $6x - 5x$, (f) $4 \times \frac{y}{2}$,
(g) $2 \times \frac{5z}{2}$, (h) $8k \div 6$,
(i) $8r - 2r - r$, (j) $4x \times 0$,
(k) $ab + ab$, (l) $aaa + aa$,
(m) $8a + a + 5a - a$, (n) $5x + 5x - x$.

18. Simplify:

(a) $s \times 1$, (b) $s \times 0$, (c) $0 \times s$, (d) $s \div 1$, (e) $1 \div s$.

19. A bag weighs 6 oz. If w lb. of groceries and z lb. of meat are put into it, what is the total weight?

20. How many apples will each of 8 boys receive if they share equally the contents of two baskets containing c apples and d apples respectively?

21. If k apples are to be shared equally between y boys and z girls, how many will each receive?

22. If a stairway has 13 risers each h inches high, what is the height of the stairway?

23. A pail holds x pints. How many times can it be filled from a tank which holds k gallons?

24. A screw has k threads and the pitch of each thread is n inches. What is the length of the threaded part?

25. Take the number n. Square it and add 7. What is the result?

26. Take the number w. Divide it by 2 and increase the result by 7. What is the new number?

27. Take the two numbers x and y. Indicate three times their sum.

28. Take the two numbers x and y. Indicate their product decreased by their sum.

29. Take the two numbers x and y. Indicate twice their product increased by 5 times their sum.

30. Make a dimensioned sketch to represent a triangle **ABC** in which the length of **AB** is $2x$ inches; that of **BC** $4x$ inches; and that of **AC** $3x$ inches. Using this sketch for reference, express and simplify as far as possible each of the following:

(a) the perimeter of the triangle;
(b) the difference between the length of **BC** and that of **AC**.

31. Draw a dimensioned sketch of a rectangle **ABCD** in which the length of **AB** is x inches and that of **BC** is $2y$ inches. Express and simplify as far as possible each of the following:

(a) the perimeter; (b) the area;
(c) the difference in length between **AB** and **BC**.

32. The distance between two points on a straight line is x yards. Express this distance in

(a) feet, (b) inches, (c) rods, (d) chains.

33. The length of a playing field is y metres. Express this distance in

(a) kilometres, (b) decimetres, (c) centimetres.

C

34. Indicate the average of a, b and c.

35. Indicate that the sum of a and b is to be subtracted from c.

36. Indicate that the sum of a and b is to be subtracted from the sum of c and d.

37. A boy is x years old and his mother is y years old. Indicate how much older the mother is than the boy.

38. You have 9 coins. If n of them are dimes and the rest are nickels, indicate the number of cents that you have.

39. You have 10 coins. If n of them are quarters, p are dimes and the rest are coppers, indicate the number of cents you have.

40. If the first dozen articles cost b cents each and each succeeding dozen costs one cent less per article, what would be the total cost of 3 dozen?

41. What is the simple interest on P dollars for t years at 5% per annum?

42. What is the simple interest on P dollars for t years at r% per annum?

43. A man who earns $\$a$ per hour has his wages increased by 10%. What is his new hourly rate? How much would he earn in t hours?

44. A box is b inches long, c inches wide and d inches deep. Find
(a) its volume, (b) the total area of all its outside surfaces,
(c) the total length of all its edges.

45. If 1 inch were added to each dimension in question 44, find expressions for
(a) the new volume, (b) the new area, (c) the new edge length.

Variables and Constants

Thus far in our discussion of measures we have considered the quantity to be a fixed size. Now let us consider a situation such as the following:

Suppose, while sitting beside the driver of a moving automobile, we attempt to write down the measure of the number of miles the car has travelled. Before we finish writing the number, it has changed. As the car moves along, the measure of the distance is continually changing.

If we use x to stand for the number of miles the car has travelled, the symbol x then stands for any one of many numbers. In this example these numbers are measures of the distance travelled. When a symbol, such as x, is used in this way it is called a *variable*. The number which the variable represents at any particular time is called a "value" of the variable. Furthermore this number for which the variable "holds a place" is a *constant*.

A symbol which serves as a "place-holder" for one or more particular numbers is a variable.

This idea that a variable can be used as a place-holder for a selected set of constants can be extended to fields of study other than mathematics.

The formula, $p = 2(b + h)$, expresses the relationship between the measure of the perimeter and the measures of the length and the width of any rectangle.

The symbols b, h and p serve as "place-holders" for the measures of length, height and perimeter respectively of any one of all possible rectangles. Thus b, h and p are variables in this relationship. Each "value" of b, h and p respectively for a particular rectangle, is a constant.

One advantage of having a knowledge of algebra is that many relationships discovered by others and expressed as formulas can be understood and used by beginners. For example, the formula stating the relationship between the measure of the area of the curved surface of any sphere and the measure of its diameter is

$$A = \pi d^2$$

where the area in square units is represented by A and the diameter in linear units by d.

This formula can be used to find the area of any sphere when the diameter is known.

Example: Find the area of a sphere the diameter of which is known to be 10 inches.

Solution: The relationship expressed by the formula is $A = \pi d^2$. The area of the sphere in square inches is $\pi \times 10^2$

$$= 3{\cdot}14 \times 100$$
$$= 314.$$

To find the area of this sphere it was necessary to substitute 10 for d in the expression πd^2.

Substitution

Example 1: What is the numerical value of $2x + 3$ which corresponds to each of the following values of x:

(*a*) $x = 0$, (*b*) $x = 1$, (*c*) $x = 3{\cdot}4$?

Solution: (*a*) When $x = 0$, $2x + 3 = 2 \times 0 + 3$
$= 0 + 3$
$= 3.$

(*b*) When $x = 1$, $2x + 3 = 2 \times 1 + 3$
$= 2 + 3$
$= 5.$

(*c*) When $x = 3{\cdot}4$, $2x + 3 = 2 \times 3{\cdot}4 + 3$
$= 6{\cdot}8 + 3$
$= 9{\cdot}8.$

To simplify an expression after substitution has been made, apply the rules for the order of operations.

Example 2: When $x = 3$ find the value of (*a*) $5x^2$, (*b*) $(5x)^2$.

Solution: (*a*) $5x^2 = 5 \times 3^2$
$= 5 \times 9$
$= 45.$

(*b*) $(5x)^2 = (5 \times 3)^2$
$= 15^2$
$= 225.$

Note the difference in meaning between $5x^2$ and $(5x)^2$.

Example 3: When $a = 5$ and $b = 4$, find the value of $6ab - 5b^2 + 2a$.

Solution: $6ab - 5b^2 + 2a = 6 \times 5 \times 4 - 5 \times 4 \times 4 + 2 \times 5$
$= 120 - 80 + 10$
$= 50.$

Exercise

A

1. If $b = 4$, what is the value of
(*a*) $2b$, (*b*) b^2, (*c*) $3b$, (*d*) b^3?

2. If $x = 3$, what is the value of
(*a*) x^3, (*b*) $4x$?

3. If $x = 5$, what is the value of
(*a*) $2x$, (*b*) $2x^2$, (*c*) $2x^3$, (*d*) $(2x)^2$, (*e*) $(2x)^3$?

4. If $x = 6$ and $y = 3$, what is the value of

(*a*) $x + y$, (*b*) $x - y$, (*c*) $x^2 + y$,
(*d*) $x^2 - y^2$, (*e*) $x^2 + y^2$, (*f*) $(x + y)^2$?

5. If $a = 12$, what is the value of

(*a*) $3a$, (*b*) $\frac{1}{2}a$, (*c*) $\frac{1}{3}a$,
(*d*) $\frac{12}{a}$, (*e*) $\frac{36}{5a}$, (*f*) $\frac{2a + 6}{a}$?

6. What is the value of x^3 for each of the following values of x:

(*a*) $x = \frac{1}{4}$, (*b*) $x = \frac{1}{2}$, (*c*) $x = \frac{3}{4}$,
(*d*) $x = 0{\cdot}1$, (*e*) $x = 0{\cdot}2$, (*f*) $x = 0{\cdot}5$?

7. If $p = 7$, $q = 8$ and $r = 0$, what is the value of each of the following:

(*a*) $p + q + r$, (*b*) $q - p + r$, (*c*) $pq - qr$, (*d*) $pq - qr$?

8. If $a = 5$, $b = 3$, $c = 0$, find the value of

(*a*) $2a^2$, (*b*) $5b$, (*c*) $a + b + c$, (*d*) $5c$,
(*e*) $2ab$, (*f*) $6bc$, (*g*) abc, (*h*) $(a + b)^2$,
(*i*) $\frac{1}{a} + \frac{b}{a}$, (*j*) $\frac{a}{b}$, (*k*) $2b^2 - 3a$, (*l*) $c(a - b)^2$.

9. If $a + b + c = 12$, what is the value of c when $a = 3$ and $b = 5$?

10. If $A = ab$, find the value of A when $a = 9$ and $b = 11$.

B

11. What is the value of $2a + 3xy$ for each of the following sets of values:

(*a*) $a = 4$, $x = 2$ and $y = 3$; (*b*) $a = 0{\cdot}3$, $x = 0{\cdot}5$ and $y = 0{\cdot}2$;
(*c*) $a = \frac{1}{2}$, $x = \frac{1}{3}$ and $y = \frac{1}{2}$; (*d*) $a = 3{\cdot}2$, $x = 1{\cdot}7$ and $y = 5{\cdot}4$?

12. Find the value of $\frac{xy}{z}$ when

(*a*) $x = 4$, $y = 5$ and $z = 2$; (*b*) $x = \frac{1}{2}$, $y = \frac{1}{3}$ and $z = \frac{1}{4}$;
(*c*) $x = 0{\cdot}2$, $y = 0{\cdot}3$ and $z = 0{\cdot}4$; (*d*) $x = 3\frac{1}{2}$, $y = 2\frac{1}{4}$ and $z = \frac{5}{8}$.

13. If $s = 16t^2$, find the values of s corresponding to the integral values of t from 1 to 6 inclusive.

14. If $A = \frac{1}{2}bh$, find the value of A when $b = 16$ and $h = 7$.

15. If $I = prt$, find the value of I when $p = 200$, $r = 6\%$ and $t = 2$.

16. If $h = \sqrt{a^2 + b^2}$, find the value of h when $a = 3$ and $b = 4$.

17. If $s = \sqrt{ab}$, find the value of s when $a = 12$ and $b = 3$.

18. If $A = 4\pi r^2$, find the value of A when $\pi = 3{\cdot}14$ and $r = 3$.

19. Find the value of $a(b - 3)$ when $a = 7$, $b = 10$.

20. Find the value of $5y(3a + 7c)$ when $y = 3$ $a = 1$, $c = 2$.

21. If $x = 3$, $y = 2$, $z = 4$, $w = 5$, find the value of:

(*a*) $x + yz + w$, (*b*) $x + y(z + w)$, (*c*) $(x + y)(z + w)$,
(*d*) $x + (y + z)w$, (*e*) $(x + y)z - w$, (*f*) $xy(z + w)$.

22. If $a = \frac{1}{2}$ and $b = \frac{1}{3}$, find the value of:

(*a*) $a - b$, (*b*) ab, (*c*) $\dfrac{a}{b}$,
(*d*) $\dfrac{1}{a}$, (*e*) $\dfrac{1}{a} + \dfrac{1}{b}$, (*f*) $9ab^2$.

23. If $x = y^2$ and $y = 5$, what is the value of x?

24. If $P = abc$ and $a = 3\frac{1}{2}$, $b = 3\frac{1}{3}$, $c = 1$, what is the value of P?

25. If $x + y = 13$ and $y = 5$, what is the value of $x - y$?

26. If $y = x^2 + 3x - 2$ and $x = 4$, what is the value of y?

27. If $y = 3x^2 + 8x - 8$ and $x = 2$, what is the value of y?

28. If $s = \sqrt{A}$ and $A = 49$, what is the value of s?

C

29. If $ab = 3$ and $a = 2$, find the value of $2(a + b)^2$.

30. If $y = x^2$ and $y = 1{,}296$, find the value of $y - x$; $x^2 - y$.

31. The square of $x + \dfrac{1}{x}$ is $x^2 + 2 + \dfrac{1}{x^2}$. Show that this is true when $x = 1$ and when $x = 4$.

32. If $x = 2y$ and $y = 5w$, find the value of xy when $w = \frac{1}{3}$.

33. When $x = 2$, find the value of

(*a*) $x^2 + 3x + 2$, (*b*) $(x + 2)(x + 1)$,
(*c*) $3x^2 + 5x + 2$, (*d*) $(3x + 1)(x + 2)$,
(*e*) $x^2 + 7x - 8$, (*f*) $(x + 8)(x - 1)$.

34. When $x = 3$, find the value of each expression in question 33. What conclusion is suggested regarding $x^2 + 3x + 2$ and $(x + 2)(x + 1)$?

The Language of General Number

Any measures such as $2x$, $x + 3$ and $x^2 - 5x + 7$ are called **expressions.** Single measures such as 2, x, or products of these such as $2x$, x^2, etc., which are combined by addition or subtraction to form an expression, are called **terms.** In the expression $5x^2 - 2x + 7$, the terms are $5x^2$, $2x$ and 7 respectively.

When a term expresses the product of two factors, each factor may be considered as the **coefficient** of the other. Usually the term coefficient refers to the numerical factor in the term. In the expression $3a + b + 6xy$, the **numerical coefficients** of the terms are 3, 1 and 6 respectively.

It is necessary to distinguish between the meaning of term and factor. Confusion results when these words are used with the wrong meaning. **Factors** of an expression are always indicated by multiplication. In $3ab$, the

factors are 3, a and b but in the expression $3 + a + b$, the numbers 3, a, and b are **terms** of the expression.

Simplification of Expressions Containing Like Terms

When two or more terms in an expression have the same literal factors, they are called **like terms.** In an expression such as $5ab - 2bc + 3ab - 7ba$, the terms $5ab$, $3ab$ and $7ba$ have the same literal factor ab; hence $5ab$, $3ab$ and $7ba$ are like terms. In an expression such as $5x^2 - 3x + 5 - 2x^2 + 6x - 4$, the terms $5x^2$ and $2x^2$ are like; $3x$ and $6x$ are like; 5 and 4 are like. Since, however, $5x^2$ and $3x$ have not the same literal factor, they are **unlike terms.**

When like terms occur in an expression they can be combined by addition and subtraction.

Example: By combining like terms, simplify each of the following expressions:

$(a)\ 3x + 5y + 2x - 3y, \qquad (b)\ 3x^2 + 3x - 5x + x^2.$

Solution:

$(a)\ 3x + 5y + 2x - 3y$
$= 3x + 2x + 5y - 3y$
$= 5x + 2y.$

$(b)\ 3x^2 + 3x - 5x + x^2$
$= 3x^2 + x^2 + 3x - 5x$
$= 4x^2 - 2x.$

Exercise

A

1. State the simplest factors of:

$(a)\ 12, \quad (b)\ 3ab, \quad (c)\ 5xy, \quad (d)\ 3x^2,$
$(e)\ 4y^3, \quad (f)\ a^2b, \quad (g)\ 3yz^2, \quad (h)\ 5m^2n^2,$
$(i)\ 6n^3, \quad (j)\ 2abc^2, \quad (k)\ (3x)^2, \quad (l)\ (2x^2)^2.$

2. State the coefficient of x in each of the following:

$(a)\ 3x, \quad (b)\ ax, \quad (c)\ 2xa, \quad (d)\ 5abx,$
$(e)\ x + 4, \quad (f)\ y + x, \quad (g)\ ax + by, \quad (h)\ x(a + b).$

3. State three expressions containing (*a*) 2 terms, (*b*) 3 terms.

4. Simplify:

$(a)\ x + 8 - 4, \quad (b)\ 5x + 9 - 2x - 7, \quad (c)\ 6x - 3 - 4x,$
$(d)\ 12 - x + 6, \quad (e)\ 10 + 5x + 10 - 5x, \quad (f)\ x + y - x - y.$

5. The length and width of a rectangle are expressed in inches by the measures x and y respectively. Express the measure of the perimeter in inches as an expression having

(*a*) four terms, (*b*) two terms, (*c*) one term.

6. The lengths of the sides of a triangle expressed in inches are represented by $3x$, $5y$ and $6x$ respectively. Express the measure of its perimeter.

7. The lengths of the sides of a quadrilateral in inches are $5x$, $8y$, $3x$ and $4y$. Express the measure of its perimeter.

8. The length of each of the equal sides of an isosceles triangle is $6x$ inches and the length of the base is $5x$ inches. Find its perimeter.

9. If the length of each side of an equilateral triangle is $13a$ feet, find
(a) its perimeter, (b) its semi-perimeter.

10. Which of the following pairs of terms are like and which are unlike:
(a) $5ab$ and $5bc$, (b) $3abc$ and $4cab$,
(c) $2a^2b$ and $3ba^2$, (d) $7a^2bc$ and $7a^2b^2c$,
(e) $4abd$ and $5dab$, (f) $6ar^2$ and $4r^2a$,
(g) $3x^2y^2z^2$ and $4x^2y^3z^2$, (h) $7a^2cd$ and $4cba^2$?

B

11. Simplify:
(a) $5a^2 + 3a - 7 + 4a^2 - 2a - 2$, ($b$) $3x + 8y + 7z - x - 3y - 2z$,
(c) $5x + 9 + 8x - 3 - 4x - 1$, (d) $2ab + 7bc - ab + 6ab - 5bc$,
(e) $4x^2 - x + 10 - 3x^2 + 8x - 7$, ($f$) $5a - b - c + a + 7b + c$.

12. If $x = 5$, find the value of $7x^2 + 6x + 9 - 5x^2 - 3x$.

13. If $x = \frac{1}{2}$, find the value of $4x^2 + 8x - 3x^2 - 7x$.

14. If $x = 2$, find the value of the sum of $x^2 - 4x + 9$ and $3x^2 - 5x + 3$.

15. Simplify and arrange in descending powers
(a) $y^3 + y^2 + y + 1 + y + y^3 + y$,
(b) $1 + 5r + r^2 - r + 7$,
(c) $5x^2 + 3x^2 + 6x^3 - 2x^2 - 6x + 3$,
(d) $3x^2 - 7x + 8 - 2x^2 - 5$,
(e) $\frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{2} + \frac{3}{4}x^2 + \frac{5}{8}x + \frac{1}{16}$.

16. The length of each side of a square is x inches; the length of each side of an equilateral triangle is x inches. What is the sum of their perimeters? What is the difference of their perimeters?

17. One square has an area of $16a^2$ square inches; another has an area of $25a^2$ square inches. What is their combined area?

18. A rectangular box has six sides. The areas of these sides in square inches are respectively $6a^2$, $8a^2$, $12a^2$, $8a^2$, $6a^2$ and $12a^2$. What is the total area of the six sides?

19. The base of a prism is a square, each side of which is x inches long. If the height of the prism is h inches, express the area of
(a) the base, (b) each side, (c) the entire surface.

20. Simplify:
(a) $a + 3a - 5ab + 3b - b$, (b) $5ab + 3ba - 20ab$,
(c) $3a^2b + 2ab^2 + 4a^2b$, (d) $3bc - 5ab - 4bc + 6ba$,
(e) $2x^2 - 3x^2 + 5x^2y + 7x^2y - 3xy^2$,
(f) $7xy - 2yz + 5x^2y + 6yz - 3xy$.

21. (*a*) Express 3′ 3″ as inches.
(*b*) Express x ft. x in. as inches.
(*c*) Express b lb. b oz. as ounces.

C

22. (*a*) When $x = 2$, does $3x + 2$ have the same value as $5x - 2$?
(*b*) When $x = 4$, does $3x + 2$ have the same value as $5x - 2$?

23. (*a*) Show that $x^2 + 8x - 7$ and $x^2 + 6x + 9$ have the same value when $x = 8$.
(*b*) Show that the difference of the values of the expressions in part (*a*) is greater when $x = 5$ than it is when $x = 7$.

24. Consider the expression $x^2 - 6x + 8$. Find its value for each integral value of x from 0 to 6. For what value of x is the expression equal to zero? As x changes from 0 to 6, when does the expression have its greatest value; its least value?

25. By using the integral values of x between 0 and 10, find a value of x which will cause the expression $x^2 - 5x + 6$ to be zero.

26. For what integral value of x will the expression $4x - 8$ be equal to zero?

Review Exercise

A

1. The area of a surface is expressed as y square yards. Express this area in (*a*) square feet, (*b*) square rods.

2. The capacity of a tank is expressed as x gallons. Express this capacity in (*a*) quarts, (*b*) pints.

3. The area of a sheet of paper is z square centimetres. Express this area in (*a*) square millimetres, (*b*) square decimetres.

4. A flask has a capacity of y litres. Express this volume in (*a*) cubic centimetres, (*b*) gallons, (*c*) pints.

5. The volume of a tank is x cubic feet. What is its capacity in gallons?

6. Express in the simplest form:

(*a*) $n + p + n + p$, (*b*) $n + n + n - n$,
(*c*) $aa + aa + aa$, (*d*) $xxx + xxx + xx + x$,
(*e*) $3x - 4x + 7x$, (*f*) $2ab + 3ba + 2a + 3b$.

B

7. What is the numerical value of the following:

(*a*) $2 + 4 \times 3$, (*b*) $2(6 + 3)$, (*c*) 2×5^2,
(*d*) $10^2 - 8^2$, (*e*) $\sqrt{13^2 - 5^2}$, (*f*) $12 \div 4 \times 3$?

8. If $x = 8$, find the value of:

(*a*) $5x$, (*b*) $3x^2$, (*c*) $\sqrt[3]{8x}$,

(*d*) $(2x)^2$, (*e*) $2x^2 - 5x + 10$, (*f*) $\frac{2x^2}{24}$.

9. When $a = 1$, $b = 2$, $c = 3$ and $d = 4$, find the numerical value of each of the following expressions:

(*a*) $2ab + 5bc + 3ac$, (*b*) $3abcd$,

(*c*) $8abc - bcd$, (*d*) $2b(5a + 4c - 2d)$,

(*e*) $5a + 3b^2$, (*f*) $5a^2b^2c + 3b^2d - 2ac^2$,

(*g*) $\frac{a}{b} + \frac{b}{c} + \frac{c}{d}$, (*h*) $\frac{a+b}{c+d} + \frac{a+d}{b+a}$,

(*i*) $(a + c)(b + d)$, (*j*) $(d - a)(b + c)$.

10. By combining like terms, simplify:

(*a*) $6x^2 - 3x + 7 - 2x^2 + 4x - 4$,

(*b*) $3xy - 5xy + 7yx + 2xy$,

(*c*) $5a^2b - 3ab^2 + 7a^2b - 6a^2b$.

CHAPTER 10

Signed Numbers

Directed Measurements

Up to this point all measures of quantities could be expressed by using integers, fractions, mixed numbers and general numbers. We have been able to solve all the problems so far by naming the quantity, the unit of measurement and the measure. Sometimes it is necessary to qualify the measure by indicating its **direction** or **sense.**

Consider each of the following statements:

(*a*) Town A has a latitude of 44°.
(*b*) North High moved the ball 8 yards on the first down.
(*c*) The ammeter reading changed 10 amperes.
(*d*) The temperature changed 15°.

In each of the above situations the measurement can be made in one of two opposite directions. Therefore, to clarify the meaning, the direction of the measurement must be stated. In mathematics, the direction of the measurement is indicated by a positive or a negative sign placed in front of the measure. **For any measurement, the direction which is to be considered positive is selected according to custom or choice; then, a similar measurement in the opposite direction is negative.**

For example, if North latitude is considered positive, then South latitude would have to be negative. If town A is north of the equator its latitude could be $+44°$. A change of temperature of $-10°$ would mean the temperature fell 10°. Signed numbers indicate the direction as well as the magnitude of the measurement.

Signs of Quality and Signs of Operation

The same symbols $+$ and $-$ that are used to indicate the operations of addition and subtraction are used to indicate the *direction* or the *quality* of the measurement. When the signs are used to indicate that the numbers are to be added or subtracted, they are called **signs of operation.** When they are used to indicate the direction of the measure, they are called **signs of quality.** The **absolute value** of the measurement is the measure without regard to direction.

In general, to describe a measurement completely, it is necessary to state

the **quantity,** the **unit of measurement,** the **sign of quality** and the **absolute value** of the measurement.

Exercise

A

1. What would a negative measurement indicate if a positive measurement meant:
   (*a*) the distance a car travelled north;
   (*b*) a rise in temperature;
   (*c*) a gain in a business transaction;
   (*d*) a deposit made in a bank account;
   (*e*) a decrease in the price of a house;
   (*f*) a loss in yards for your rugby team;
   (*g*) a force pulling downward;
   (*h*) a force pulling upward;
   (*i*) the turning of a wheel in a counter-clockwise direction?

2.

   If measurements to the right are considered positive, and if each space is 1 unit, give the directed measure of the distance from:
   (*a*) **C** to **A**, (*b*) **E** to **K**, (*c*) **K** to **F**,
   (*d*) **Q** to **C**, (*e*) **F** to **C**, (*f*) **Q** to **M**,
   (*g*) **F** to **E**, (*h*) **M** to **C**, (*i*) **B** to **A**.

3. The balance in a bank account at the beginning of a month is \$15. What will be the balance after entries have been made as follows:
   (*a*) \$+5, \$−8, \$−2, \$+7; (*b*) \$−10, \$+5, \$+20;
   (*c*) \$−1, \$−7, \$+6; (*d*) \$−8, \$−9, \$+70?

4. The price of a certain stock at the opening of the market on Monday was given as 58. The daily fluctuations in price were as follows: +2, −3, +5, −1, −2. What was the closing price on Friday?

B

5. During one week a firm carried out transactions with results as indicated below: On Monday it gained \$50; on Tuesday it lost \$10; on Wednesday it gained \$75; on Thursday it neither gained nor lost; on Friday it lost \$25; on Saturday it gained \$45.
   (*a*) Express each day's *profit* using a signed number.
   (*b*) What was the net profit for the week's transactions?
   (*c*) What is the sum of the signed numbers +50, −10, +75, −25, +45?
   (*d*) What is the sum of the signed numbers −2, +5, −6?

6. What is the sum of
   (*a*) −3, −4, −7; (*b*) −8, −7, −3?

7. What is the sum of the following signed numbers:

(*a*) $+5, +8, +7$; (*b*) $+10, +8, +9$?

8. Find the sum of the following signed numbers:

(*a*)	(*b*)	(*c*)	(*d*)	(*e*)
$+5$	-3	$+8$	-8	-3
$+2$	-8	-5	$+6$	$+7$

9. Add:

(*a*)	(*b*)	(*c*)	(*d*)
-2	$+7$	$+3$	-3
-3	$+4$	-5	$+5$
(*e*)	(*f*)	(*g*)	(*h*)
$+2x$	$-5x$	$+4x$	$-3a^2$
$+3x$	$-4x$	$-7x$	$+8a^2$

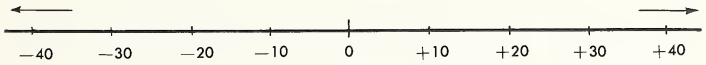
The Signed Number System

From any starting-point on a horizontal line there are two possible directions in which to move along the line. A move to the right will be considered positive; then a move to the left is negative.

Consider the starting-point as 30. Then a move of $+10$ would be a move to the right and we would arrive at 40; a move of -10 would be a move to the left and we would arrive at 20.

If, however, the starting-point on the line were taken as zero, then a move of $+10$ would take us to the right, to a position called positive 10. A move of -10 would take us to the left to a position which we now must call negative 10. Similarly, any other negative move would bring us to a position which is as far to the left of zero as a corresponding positive move is to the right.

The central portion of the signed number system can be represented by distances measured along a line in both directions from a reference point. This reference point is called the **origin.**



Positive numbers are represented by distances to the right of the origin. Then, of course, negative numbers are represented by distances to the left of the origin.

Signed numbers have been used here for two purposes

(*a*) **to indicate a move in a certain direction;**

(*b*) **to indicate a position on a scale of numbers using the origin as the dividing-point between positive and negative numbers.**

Line Representation of Signed Numbers

Distances along a line have been used to represent arithmetical numbers. A similar line representation of literal numbers will help us to become better acquainted with general numbers.

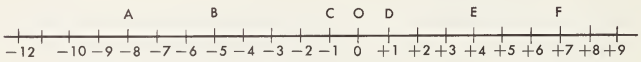
On the line representation above, both positive and negative numbers are shown. Some position to the right of zero represents a number x. The position 5 units to the right of x therefore indicates the number $x + 5$. Similarly, the position 5 units to the left of x indicates the number $x - 5$.

Since the position representing y on this diagram is to the left of zero, then y represents a negative number. Numbers larger or smaller than y would be represented as indicated.

Exercise

A

1. State the number represented by the position of each letter in the diagram:



2. (*a*) Which of the signed numbers in Question 1 is the greatest?
(*b*) Which of the signed numbers in Question 1 is the least?

3. What signed number will indicate the move from
(*a*) **A** to **B**, (*b*) **A** to **C**, (*c*) **C** to **D**, (*d*) **D** to **E**,
(*e*) **A** to **F**, (*f*) **F** to **E**, (*g*) **E** to **C**, (*h*) **E** to **A**?

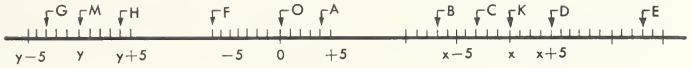
4. What number will indicate the position on the scale when $+2$ is added to $+3$?

5. Use the scale to assist you to find the sum when
(*a*) $+3$ is added to $+5$, (*b*) $+2$ is added to -3,
(*c*) $+7$ is added to -4, (*d*) $+3$ is added to -3.

6. What number will indicate the position along the scale when -2 is added to $+3$?

7. Find the sum when
(*a*) -5 is added to $+8$; (*b*) -3 is added to $+2$;
(*c*) -4 is added to -3; (*d*) -5 is added to $+5$.

8.

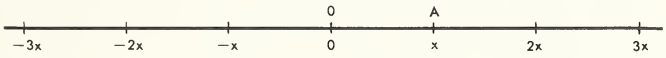


(*a*) State the number represented by each of the positions indicated by a capital letter.

(*b*) What number would be represented by each of the positions described as follows:

(i) 3 units to the right of **A**, (ii) 6 units to the left of **A**,
(iii) z units to the right of **A**, (iv) 8 units to the right of **D**,
(v) 6 units to the left of **D**, (vi) 7 units to the right of **B**,
(vii) 9 units to the left of **G**, (viii) b units to the right of **K**?

9. In the line representation that is illustrated here, the distance from the origin **O** to **A** represents the number x. Thus x is a positive number.



Some multiples of x are indicated on the diagram. Using this diagram, perform each of the following operations:

(*a*) add $+2x$ to $+3x$; (*b*) add $+2x$ to $-4x$;
(*c*) add $-2x$ to $+5x$; (*d*) add $-2x$ to $-x$;
(*e*) subtract $+3x$ from $+4x$; (*f*) subtract $+2x$ from $-3x$;
(*g*) subtract $-2x$ from $+3x$; (*h*) subtract $-3x$ from $+4x$.

10. In this line representation, each small division indicates one unit.

(*a*) State the number represented by each of the positions **A**, **B**, **C**, **D** and **E** respectively.

(*b*) What is the sum of x and $2x + 5$?

(*c*) What is the sum of $x - 5$ and $3x + 1$?

(*d*) Subtract $2x$ from $4x + 5$.

(*e*) Subtract $3x + 1$ from $5x + 6$.

B

11. Answer the following problems in sentence form:

(*a*) If n is an integer, what is the next integer greater? What is the integer preceding n?

(*b*) What is the number 25 greater than x; 25 less; 25 times as much?

(*c*) What is the next even integer above (i) 30, (ii) 44, (iii) $2k$?

12. Construct a line representation of numbers and mark a positive number y on it at some point **A**. Choose a suitable distance to represent one unit. Mark on this line the positions to indicate:

(a) $y + 2$, (b) $y - 3$, (c) $-y$,
(d) $2y$, (e) $2y + 5$, (f) $-y + 4$,
(g) $2y - 8$, (h) $3y$, (i) -3.

13. Construct a horizontal line. Choose a suitable scale and mark the following numbers on it: 0, $+3$, $+2\frac{1}{2}$, -3, -1, $-\frac{3}{4}$, $+\frac{1}{2}$, $-0{\cdot}8$, $+0{\cdot}25$.

14. On a horizontal line 6″ long, construct a scale which will represent all the numbers from $-1{,}000$ to $+500$. Mark all the exact hundreds.

15. Copy the following in your notebook and complete each sentence with the proper expression:

(a) Five more than a is .....
(b) Seven more than three times y is .....
(c) The sum of 3 times x and 4 times y is .....
(d) The sum of x, y and w is .....
(e) Five less than x is .....
(f) x increased by 10 is .....
(g) y decreased by 4 is .....
(h) x diminished by 7 is .....
(i) Fifteen diminished by 3 times b is .....
(j) If n is an odd integer, the next even integer above n is .....
(k) If n is even, the next even number above n is .....

16. On a line representation of numbers, mark a positive number z at some point **A**. Choose a suitable distance to represent one unit. Mark on this line the positions to indicate:

(a) $-z$, (b) $z + 3$, (c) $2z - 2$,
(d) $3z + 1$, (e) $4 - z$, (f) $-2z - 3$.

CHAPTER 11

Operations with Signed Numbers

Let us keep in mind the fact that, at any one time, only two numbers are involved in any operation. Of course, the operations may be continued with additional numbers. The two numbers may be

(*a*) both positive or both negative,

(*b*) one positive and one negative.

Accordingly, the two numbers may have like signs or unlike signs of quality.

Addition of Signed Numbers

Addition of Numbers with Like Signs

With the aid of line representation of numbers or by other means perform the following operations:

(*a*) Find the sum: $(+2x) + (+3x)$.

Since the terms are alike, the addition may be performed.

$$\begin{array}{r} +2x \\ +3x \\ \hline \text{The sum is } +5x \end{array} \qquad \text{or} \qquad \text{The sum of } (+2x) + (+3x) = +5x.$$

The sign of the sum of two positive numbers is positive and the absolute value of the sum is found by numerically adding the absolute values of the addends.

(*b*) Find the sum of -3 and -2.

Again the solution may be put down in either of two ways.

$$\begin{array}{r} -3 \\ -2 \\ \hline \text{The sum is } -5 \end{array} \qquad \text{or} \qquad \text{The sum of } -3 \text{ and } -2 \text{ is } (-3) + (-2) = -5.$$

The sign of the sum of two negative numbers is negative and the absolute value of the sum is found by numerically adding the absolute value of the addends.

The sum of numbers having the same sign will have

(i) *the same sign as that of its addends,*

(ii) *an absolute value which is the sum of the absolute value of its addends.*

Example 1: Find the sum of $5a + 3b - 6c$, $7a + 4b - 5c$ and $a + 2b - 3c$.

Solution:

$$\begin{array}{r} 5a + 3b - 6c \\ 7a + 4b - 5c \\ a + 2b - 3c \\ \hline 13a + 9b - 14c \end{array}$$

The sum is $13a + 9b - 14c$.

Example 2: Find the sum of $3x^2 - 2x + 5$, $8 - 3x$ and $5 + x^2$.

Solution: Rearrange the expressions to have like terms placed in the same column.

$$\begin{array}{r} 3x^2 - 2x + 5 \\ -3x + 8 \\ x^2 \quad + 5 \\ \hline 4x^2 - 5x + 18 \end{array}$$

The sum is $4x^2 - 5x + 18$.

Exercise

A

Find the sum of the following sets of signed numbers:

1. $+3, +2, +5$;
2. $+37, +49$;
3. $-2, -3, -4$;
4. $-28, -69$;
5. $-8, -6, -3$;
6. $+7, +8, +3$;
7. $+8, +5, +9, +5$;
8. $-6, -7, -8, -4$;
9. $-1, -1, -1, -2$;
10. $-12, -6, -5, -7$;
11. $+3x, +5x, +7x$;
12. $-5a, -4a, -6a$;
13. $5ab, +4ab, +6ba$;
14. $a, +c, +b, +a$;
15. $a + 1, a + 2, a + 3$;
16. $3a + 5b, 5a + 7b$;
17. $2x^2 - 4x, 5x^2 - 9x$;
18. $x^2 - 3x + 2, x^2 - 7x + 15$.

B

Find the sum of:

19. $+3{\cdot}4, +2{\cdot}7, +0{\cdot}83$;
20. $-2\frac{1}{2}, -5\frac{3}{4}, -\frac{1}{8}$;
21. $+\frac{1}{8}, +\frac{1}{2}, +\frac{3}{4}$;
22. $-\frac{7}{16}, -\frac{3}{32}, -\frac{5}{64}$;
23. $-\frac{5}{6}, -\frac{9}{10}, -\frac{8}{15}$;
24. $+\frac{5}{9}, +\frac{3}{4}, +\frac{7}{12}, +\frac{5}{6}$;
25. $-0{\cdot}56, -8{\cdot}3, -7{\cdot}2$;
26. $-0{\cdot}375, -0{\cdot}035, -0{\cdot}009$.

Arrange the terms in vertical columns and find the sum:

27. $3a + 5b - 6c, 2a + b - c, 6a + 7b - 9c$;
28. $3x - 5y + z, 5x - 8y + 3z, 2x - y + 10z$;
29. $5x^2 - 3x^3 + 4x, -7x^3 + 3x^2 + x, 8x - x^3 + x^2$;
30. $4ab + 5ac - bc, -5bc + 3ab + 7ac, ab + ac$;
31. $2x^2 - 7, x^2 - 8y - 3, 2x^2 - 7y$;

32. $x^5 + 5x^4$, $3x^5 + 7x^4$, $x^4 - x^3$;

33. $x^3 + 2x^2 + x$, $x^2 + 2x + 1$.

34. An isosceles triangle has sides of $13x$ inches, $13x$ inches and $10x$ inches. Find the perimeter.

35. An equilateral triangle has sides of $15x$ inches. Find the perimeter.

36. The circumferences of three circles are 16π inches, 20π inches and 8π inches. Find the sum of the three circumferences.

37. If the diameters of three circles are 4″, 5″ and 6″ respectively, find the sum of their circumferences.

38. The sides of a quadrilateral are $5x - 2y$ inches, $4x - y$ inches, $2x - y$ inches and $8x - 3y$ inches. Find the perimeter.

39. Add $-x - y$, $\quad -y - z$, $\quad -z - x$, $\quad -x - y - z$.

40. Add $2(a + b)$, $3(a + b)$, $4(a + b)$.

41. If $x = 4$, find the value of the sum of $x^2 - x$ and $2x^2 - 3x + 8$.

C

42. If $a = 5$, find the value of the sum of $2a^2 - 4a - 7$, $a^2 - a - 2$ and $3a^2 - 5a - 8$.

43. If $x = 7$, $y = \frac{1}{2}$ and $z = 1\frac{1}{4}$, find the value of the sum of $3x - y + 4z$, $2x - y + z$ and $x - 2y + 5z$.

44. Add $3(x - y) + 5(y - z)$, $4(x - y) + (y - z)$ and $7(x - y) + 5(y - z)$.

45. Find the sum of $(x - y) + (y - z)$ and $(x - y) + (z - y)$.

Addition of Numbers with Different Signs

By using a line representation of numbers we find that

$$(+7) + (-10) = -3;$$
$$(+2x) + (-3x) = -x;$$
$$(-10y^2) + (15y^2) = +5y^2.$$

It will be noted that *when two numbers with different signs are added, the sign of the sum is the same as that of the addend with the larger absolute value and the absolute value of the sum is the numerical difference of the absolute values.*

The rule for adding signed numbers may be stated briefly thus:

To add numbers with like signs, put down the common sign and add the absolute values.

To add numbers with different signs, put down the sign of the number with the greater absolute value and find the numerical difference of the absolute values.

Example: Find the sum of $3xy + 2yz - 5$, $3yz - 4xy - 8$ and $5 - 2yx - 7yz$.

Solution:

$$\begin{array}{r} 3xy + 2yz - 5 \\ -4xy + 3yz - 8 \\ -2xy - 7yz + 5 \\ \hline -3xy - 2yz - 8 \end{array}$$

The sum is $-3xy - 2yz - 8$

Exercise

A

Find the sum of the following sets of signed numbers:

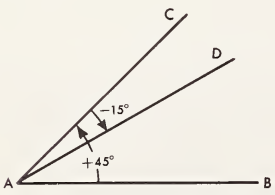
1. $+5$ and -8;
2. $+15$ and -7;
3. -84 and $+46$;
4. $+18$ and -27;
5. $+8$ and -12;
6. $+19$ and -16;
7. $+9$, $+8$, -3;
8. $+7$, $+6$, $+5$, -9;
9. -6, $+3$, -4, -2;
10. -9, -8, -3, -4;
11. $3x$, $5x$, $-7x$;
12. $-3a$, $-4a$, $+5a$;
13. $6x^2$ and $-7x^2$;
14. $2a$, $+2a^2$, $-3a$ and $-5a^2$;
15. $6x$, $-7y$, $+5x$ and $-3y$;
16. $2xy$, $+5xz$, $-8xy$, $+7xz$.

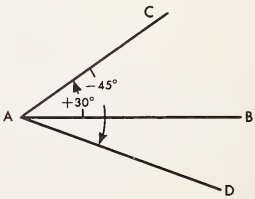
B

Find the sum of the following sets of signed numbers:

17. $+\frac{1}{4}$, $-\frac{1}{2}$ and $+\frac{1}{2}$;
18. $-2\frac{3}{4}$ and $+5\frac{1}{2}$;
19. $+3\frac{1}{2}$ and $-2\frac{1}{4}$;
20. $-\frac{7}{16}$, $+\frac{5}{32}$ and $+\frac{3}{8}$;
21. $-0{\cdot}5$, $+0{\cdot}03$ and $-0{\cdot}25$;
22. $-3{\cdot}0$ and $+2{\cdot}5$.

23. The measurement of an angle is usually considered to be positive when the line rotates in a counter-clockwise direction from its initial position. Then of course, when the line rotates in a clockwise direction the measurement of the angle would be negative.





What angle will a line make with its initial position when it has rotated through

(*a*) $+45°$ and then through $-15°$;
(*b*) $+30°$ and then $-45°$;
(*c*) $-10°$, $+20°$ and then $+50°$;
(*d*) $+120°$, $-40°$, $+200°$, $-10°$?

Arrange in columns and find the sum:

24. $3a - 2b + 6c$, $5a - b + 7c$, $-3a + 6b - 8c$;

25. $4x^2 - 8x + 7$, $9x - 3x^2 + 10$, $-x + 7 - x^2$;

26. $-1 + 8c - a$, $-5 - 3c + 4a$, $b - 4c - 4a$;

27. $x^3 - 3x^2 + 6x$, $+6x^2 + 9x - 18$;

28. $x^3 - 9$, $x^3 - 1$, $x^3 - 27$, $x^3 - 4$, $x^2 - 9$;

29. $4ab + 5b - 6c$, $4b + 8c - 7ab$, $5c - b + 7ab$.

30. The four sides of a quadrilateral are $x + y$, $x - y$, $2y - x$ and $5x + 3y$. Find the perimeter.

31. If $x = 1$, find the value of the sum of $x^2 + 1$, $x^2 - 6$, $5x^2 + 4$ and $2x^2 - 3x + 1$.

32. If $x = 0$, find the value of the sum of $3xy + 6$, $5x^2 - 3$, $4yzx - 10$ and $8 - 3x$.

C

33. If $x = \frac{1}{2}$, find the numerical value of the sum of $2x^2 - 3x + 8$, $5x^2 - 27 + 4x$ and $18 - 6x - 3x^2$.

34. Find the sum of the following consecutive numbers: $3n - 3$, $3n - 2$, $3n - 1$, $3n$, $3n + 1$, $3n + 2$, $3n + 3$ and $3n + 4$.

35. Find the sum of three consecutive numbers of which n is the middle one.

36. Find the sum of 5 consecutive numbers of which x is the least.

37. Find the sum of four consecutive even numbers of which $2x - 6$ is the greatest.

38. Find the sum of five consecutive odd numbers of which $2x + 1$ is the middle one.

39. Find the sum of $3{\cdot}5a + 6{\cdot}7b - 3{\cdot}2c$ and $0{\cdot}4a - 5{\cdot}4b + 6{\cdot}3c$.

40. Simplify $1{\cdot}17a + 8{\cdot}56a - 3{\cdot}46a - 9{\cdot}15a$.

41. Find the sum of $3(a + b) - 7(a - c)$, $6(a - c) - 5(a + b)$ and $4(a + b) - (a - c)$.

42. Find the sum of $3(x + y) + 5(y - z)$, $4(x + y) - (y - z)$ and $-9(y - z) - (x + y)$.

Subtraction

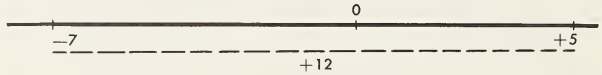
Since subtraction is the inverse of addition, to find the difference when $+5$ is subtracted from $+7$ we may ask "*What must be added to* $+5$ *to give* $+7$?" The answer is $+2$.

$$\therefore (+7) - (+5) \text{ is } +2.$$

To subtract -7 from $+5$, we find the number which must be added to -7 to give $+5$. The number is $+12$.

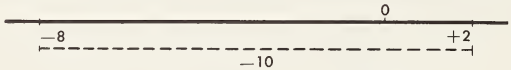
$$\therefore (+5) - (-7) = +12.$$

Let us consider this problem from the point of view of line representation of numbers. The number to be added is the distance from the subtrahend to the minuend with the proper sign affixed to indicate the direction.



$$\therefore (+5) - (-7) = +12.$$

In a similar way we may find the difference when +2 is subtracted from −8.



From +2 to −8 is 10 units in the negative direction.

Subtract −8
+2 or $(-8) - (+2) = -10$

Difference is −10.

Exercise

A

By considering that subtraction is the inverse of addition, find the difference when

1. +8 is subtracted from +10; **2.** −5 is subtracted from +3;

3. +6 is subtracted from +2; **4.** +3 is subtracted from −5;

5. −8 is subtracted from −6.

By comparing with positions on a number scale, find the difference when

6. +3 is subtracted from +7; **7.** −4 is subtracted from +2;

8. +5 is subtracted from +3; **9.** +2 is subtracted from −5;

10. −7 is subtracted from −3.

11. Using signed numbers, express the change in temperature for each of the following:

Final temperature in degrees	+25	+25	+25	0	0
Initial temperature in degrees ...	+15	+30	−15	+5	−10
Change in temperature in degrees					

12. What number must be added to

(*a*) +10 to give +15, (*b*) −5 to give +10,

(*c*) −12 to give −8, (*d*) +6 to give +2,

(*e*) +5 to give 0, (*f*) −5 to give 0?

13. What is the excess of

(*a*) $+25$ over $+20$, (*b*) $+5$ over -10,
(*c*) $+5$ over $+10$, (*d*) -3 over -7,
(*e*) -8 over $+15$, (*f*) $+3\frac{1}{4}$ over $7\frac{1}{8}$?

14. How much greater is

(*a*) $+25$ than $+24$, (*b*) $+15$ than $+20$,
(*c*) $+10$ than -5, (*d*) -10 than $+5$?

15. Subtract:

(*a*) $5x$ from $-3x$, (*b*) $8y$ from $5y$,
(*c*) $-5a$ from $6a$, (*d*) $8a$ from $-2a$,
(*e*) $-8a$ from $+2a$, (*f*) x^2 from $-3x^2$.

16. What is the excess of:

(*a*) $x + 7$ over x, (*b*) $y + 2$ over $y + 7$,
(*c*) $3x + 8$ over $2x + 5$, (*d*) $2x - 7$ over $3x + 4$?

17. Find the value of:

(*a*) $(+6) - (-5)$, (*b*) $(+6) - (+5)$, (*c*) $(+6) + (-5)$,
(*d*) $(+6) + (+5)$, (*e*) $(-6) - (+3)$, (*f*) $(+6) - (-3)$.

18. Find the value of:

(*a*) $(4a) - (-5a)$, (*b*) $(4a) - (+4a)$, (*c*) $0 - (-4a)$,
(*d*) $0 - (+4a)$, (*e*) $a - a$, (*f*) $6a - 8a$.

B

19. By a method of your own choosing, perform the subtraction indicated and verify your result by addition:

(*a*) $+7x$ (*b*) $+3a^2$ (*c*) $+2xy$ (*d*) $-5yz$ (*e*) $-2xz$
$+4x$ $-5a^2$ $+7xy$ $-8yz$ $+5xz$

20. What must be added to $3a + 4b$ to give $7a + 8b$?

21. What must be added to $a + 9b$ to give $7a + 8b$?

22. What must be added to $-5a + 7b$ to give $-8a - 7b$?

23. What must be added to $6x - y$ to give $x - y$?

24. Perform the following subtractions:

(*a*) $3x - 5b$ (*b*) $3x^2 - 5x + 8$ (*c*) $3a - b - 6c$
$4x - 8b$ $5x^2 - 3x + 4$ $a + 3b + 6c$

Rule for Subtraction of Signed Numbers

All subtractions may be performed by reference to the line representation of numbers but a further observation shows that there is a shorter device for obtaining differences between signed numbers.

Study the following pairs of illustrations:

$$(+3) - (-5) = +8$$
$$(+3) + (+5) = +8$$

$$(+5x) - (+10x) = -5x$$
$$(+5x) + (-10x) = -5x$$

$$(-7a) - (-10a) = +3a$$
$$(-7a) + (+10a) = +3a$$

In each case it will be noted that the subtraction could have been performed by changing the sign of the subtrahend and adding it to the minuend.

To subtract signed numbers, change the sign of the subtrahend and add the resulting signed numbers.

Example 1: Subtract -8 from $+15$.

Solution:

$$\begin{array}{rr} \text{Subtract} & +15 \\ & \underline{(+)8} \\ & +23 \end{array} \qquad \text{or} \qquad \begin{array}{l} (+15) - (-8) \\ = (+15) + (+8) \\ = +23 \end{array}$$

Example 2: Subtract $5x - 7y$ from $2x - 3y$.

Solution:

$$\begin{array}{rr} \text{To subtract} & 2x - 3y \\ & 5x - 7y \end{array}$$

Change the signs of the terms in the subtrahend and add.

$$\begin{array}{r} 2x - 3y \\ \underline{-5x + 7y} \\ -3x + 4y \end{array}$$

Example 3: What is the excess of $5x^2 - 2x + 5$ over $2x^2 + 5x - 8$?

Solution:

$$\begin{array}{rr} & 5x^2 - 2x + 5 \\ & \underline{2x^2 + 5x - 8} \\ \text{The excess is} & 3x^2 - 7x + 13 \end{array}$$

Check:

$$\begin{array}{rr} & 2x^2 + 5x - 8 \\ & \underline{3x^2 - 7x + 13} \\ \text{The sum is} & 5x^2 - 2x + 5 \end{array}$$

Exercise

A

1. Subtract:

(*a*) $+15$ from $+20$, (*b*) -10 from $+5$, (*c*) -8 from -9,
(*d*) -7 from -5, (*e*) $+5$ from 0, (*f*) -8 from 0.

2. Perform the following indicated subtractions:

(*a*) $(+10) - (+5)$, (*b*) $(+10) - (-5)$,
(*c*) $(-15) - (+7)$, (*d*) $(+8) - (+10)$,
(*e*) $(-8) - (-6)$, (*f*) $(-6) - (-10)$.

3. Subtract:

(*a*) $+8x$ from $-7x$, (*b*) $-3a$ from $+7a$, (*c*) $2x^2$ from $5x^2$,
(*d*) $-8y$ from $3y$, (*e*) $6y$ from $2y$, (*f*) $8a$ from $10a$.

4. How much must be added to
(*a*) $8x$ to give $12x$, (*b*) $6y$ to give $-3y$,
(*c*) a to give $-a$, (*d*) x to give $-8x$?

Subtract:

5. $+8$ / -3

6. $-5x^2$ / $+9x^2$

7. $+7a$ / $-9a$

8. $-3x$ / $+5x$

9. $+9ab$ / $+11ab$

10. $-3m$ / $+5m$

11. $-8xy$ / $-5xy$

12. $-6x^2$ / $-8x^2$

13. $+8x^3$ / $+12x^3$

14. $5x - 3y$ / $2x + 4y$

15. $3x^2 + 2x$ / $5x^2 - 3x$

B

Subtract:

16. $2a + 3b$ / $a + b$

17. $4a - 6b$ / $2a - 3b$

18. $4x - y$ / $6x - 7y$

19. $5a - b$ / $4a - b$

20. $x + y$ / $x - y$

21. $4a - 7b$ / $6a + 3b$

22. $5x - 3y + 4z$ / $8x + 6y - 3z$

23. $4a + 3b - c$ / $-5a - 4b - 3c$

24. $a + b - c$ / $a + 3b - 4c$

25. $5a + 2b$ / $2a - 2b$

26. $4a + b$ / $a + 4b$

27. a / $a + b$

28. $x + 1$ / $x + 5$

29. 3 / $x - y + 5$

30. 0 / $a + b$

31. x / $x - y + 2$

32. $15x - 3y$ / $15x - 30y$

33. $x + 2y - 8$ / $5x + 2y - 18$

34. $5x - 5y - 10$ / $4x - 5y + 8$

35. $0{\cdot}2a + 0{\cdot}8b$ / $0{\cdot}8a - 0{\cdot}6b$

36. $0{\cdot}3a - 0{\cdot}1b$ / $0{\cdot}2a - 0{\cdot}7b$

37. Subtract $3a + 5b$ from the sum of $6a - 7b$ and $2a - b$.

38. Subtract the sum of $a + 3b$ and $5a - 7b$ from $a - 3b$.

39. Subtract $4x^2 - 6x - 8$ from $2x^2 + 3x - 2$ and check by addition.

40. Using signed numbers, express each of the two differences between these numbers:
(*a*) $+13$ and $+8$, (*b*) -7 and -5,
(*c*) $+11$ and -3, (*d*) $-3\frac{1}{2}$ and $+2\frac{1}{4}$,
(*e*) 0 and $+5$, (*f*) 0 and -8.

41. Find the difference when

(*a*) $-2{\cdot}35$ is subtracted from $+5{\cdot}83$;
(*b*) $+5{\cdot}76$ is subtracted from $+1{\cdot}89$;
(*c*) $0{\cdot}03$ is subtracted from $+0{\cdot}01$;
(*d*) $+0{\cdot}30$ is subtracted from $+0{\cdot}03$;
(*e*) $-0{\cdot}36$ is subtracted from $-0{\cdot}43$.

Removal of Brackets Preceded by a Plus Sign

Since most of the operations with which we shall be dealing will be expressed in horizontal form, special attention will be given to the simplification of indicated operations.

We have seen that $(-3) + (-2)$ is equivalent to $-3 - 2$ or -5.
Also $(+5) + (-3)$ is equivalent to $+5 - 3$ or $+2$. To obtain the second line of these simplifications three alterations were made; namely,

(1) the brackets were dropped;
(2) the plus sign indicating the addition was dropped;
(3) the original signs of quality were used to indicate the operations to be performed with the numbers.

The sum of $5x - 3y$ and $-2x + 9y$ may be found in either of two ways.

$$\begin{array}{rl} & 5x - 3y \\ & -2x + 9y \\ \text{The sum is} & \overline{3x + 6y}. \end{array} \qquad \text{or} \qquad \begin{array}{l} \text{The sum is } (5x - 3y) + (-2x + 9y) \\ \qquad = 5x - 3y - 2x + 9y \\ \qquad = 3x + 6y. \end{array}$$

To remove brackets preceded by a plus sign of operation, drop the plus sign of operation and keep the signs of quality to indicate the operations to be performed with the terms.

Example 1: The lengths of the sides of a triangle expressed in inches are $x - y$, $5x + 2y$ and $3x - 4y$. Find the perimeter.

Solution: The perimeter in inches is $(x - y) + (5x + 2y) + (3x - 4y)$
$= x - y + 5x + 2y + 3x - 4y$
$= 9x - 3y.$

Removal of Brackets Preceded by a Minus Sign

The difference when $+5$ is subtracted from $+2$ is indicated by $(+2)-(+5)$. If we change the sign of the subtrahend and change the operation to addition, we find that

$$\begin{aligned} (+2) - (+5) &= (+2) + (-5) \\ &= +2 - 5 \\ &= -3. \end{aligned}$$

Similarly the difference when $3x - y$ is subtracted from $2x + 3y$ is expressed by $(2x + 3y) - (3x - y)$. If we change the sign of each term of

the subtrahend and change the operation to addition, we find that

$$\begin{aligned}(2x + 3y) - (3x - y) &= (2x + 3y) + (-3x + y)\\ &= 2x + 3y - 3x + y\\ &= -x + 4y.\end{aligned}$$

To remove brackets preceded by a minus sign of operation, drop the minus sign of operation, change the signs of quality and use these as signs of operation.

Example 1: The length of a line **AB** expressed in inches is $2x - y$ and that of **CD** is $3x + 2y$. How much greater is the length of **CD** than **AB**?

Solution: The excess of **CD** over **AB** in inches is

$$\begin{aligned}(3x + 2y) - (2x - y)\\ = 3x + 2y - 2x + y\\ = x + 3y.\end{aligned}$$

Example 2: Simplify $(a - 13b) - (-4a + b) + (-2a + 5b)$.

Solution:

$$\begin{aligned}(a - 13b) - (-4a + b) + (-2a + 5b)\\ = a - 13b + 4a - b - 2a + 5b\\ = 3a - 9b.\end{aligned}$$

Exercise

A

1. For each of the following indicated additions, remove the brackets and simplify:

(*a*) $(+5) + (+6)$, (*b*) $(-5) + (-3)$,
(*c*) $(+2) + (-7)$, (*d*) $(+8) + (-2)$,
(*e*) $(+5) + (-3) + (-7) + (-2)$,
(*f*) $(-3) + (-3) + (-3) + (-3)$.

2. Simplify:

(*a*) $(+5x) + (+8x) + (+3x)$, (*b*) $(-3x) + (-5x) + (-7x)$,
(*c*) $(-7y) + (-8y) + (-y)$, (*d*) $(3ab) + (-5ab) + (8ab)$.

3. Remove the brackets and simplify each of the following:

(*a*) $(+5) - (+7)$, (*b*) $(-3) - (-8)$,
(*c*) $(+6) - (-7)$, (*d*) $(-3) - (+5)$.

4. Read with the brackets removed:

(*a*) $(a - 3b) - (2a - 4b) + (-5a - 3b)$;
(*b*) $(-2a - 3b) - (-4a - 7b) + (2a - 3b)$;
(*c*) $(5x^2 - 3x + 4) + (-3x^2 + 6x - 8)$;
(*d*) $-(5a + 3b - c) - (+8a - 3b + 6c) + (a - b + c)$.

B

5. Simplify:

(*a*) $(3x^2 + 2x - 7) + (5x^2 - 8x - 3)$;
(*b*) $(3a + 4b - 5c) + (-4a - 8b - c)$;

(*c*) $(2x + 3y) + (+5x + 6y) + (-3x - 10y)$;
(*d*) $(2a^2 - 5ab + 5b^2) + (5a^2 + 2ab + 7b^2) + (a^2 + 2ab + b^2)$.

6. The sides of a triangle expressed in inches are $3a + 4b$, $6a - 5b$ and $7b - a$. Find the perimeter.

7. Express the sums of the following numbers as indicated additions and simplify:

(*a*) $+3x + 5y$ and $-5x + 6y$, (*b*) $2x^2 + 3x$ and $-4x^2 - 6x$,
(*c*) $3a - 5b$ and $+5a + b$, (*d*) $4ab - 3bc$ and $-6bc + 4ba$.

8. Simplify:

(*a*) $(+5x) + (-5x) - (-3x)$; (*b*) $2a + (2a + 3) - (2a - 3)$;
(*c*) $(a + 3) + (a - 3) - (a + 5)$; (*d*) $(x - y) - (x + y)$;
(*e*) $(5x - 3) + (-4x - 5)$; (*f*) $(5x - 3) - (-4x - 5)$.

9. Remove the brackets and express in simplified form:

(*a*) $(3x + 4y) + (-5x + 3y)$; (*b*) $(5a - 2b) - (3a - 7b)$;
(*c*) $(5a - 2b) - (3a + 2b)$;
(*d*) $(3x - 2y + 5z) - (-2x - 5y + 3z)$.

10. Simplify:

(*a*) $(x^2 - 3x + 2) + (x^2 + 5x - 3) - (x^2 - 8x + 7)$;
(*b*) $(a^2 - 4a + 4) - (a^2 - 7a + 1) + (3a^2 - 3a + 8)$;
(*c*) $-(x^2 - 2xy + y^2) + (x^2 + y^2) - (x^2 - y^2)$;
(*d*) $(2xy - 3yz - 9zx) - (-4xy - 4zx - 3yz) - (xy - yz + zx)$.

11. What must be added to

(*a*) -7 to obtain $+8$, (*b*) $2a - 3$ to obtain $5a - 7$?

12. What is the excess of (*a*) $+8$ over -5; (*b*) $3x$ over $5x - y$?

13. How much greater is (*a*) -10 than $+5$; (*b*) $5x - 3y$ than $2x + 4y$?

14. Simplify:

(*a*) $(a - 3b) + (5a - 7b) - (2a + 4b)$;
(*b*) $(3x^2 - 5x + 7) - (2x^2 + 5x - 3) + (5x^2 - x - 1)$;
(*c*) $-(2x - 3y + 4z) + (-2x - 3y + 4z)$;
(*d*) $-(-2x - 3y) - (5x - 2y) - (3y - 5x) + (-2x - y)$.

15. Subtract $3a + 4b - 2c$ from the sum of $a + 5b + c$ and $2a - b - 3c$.

16. Subtract $2x - 3y + z$ from $-4x - y - 3z$ and to the difference add $7x - y + 3z$.

C

17. If $x = 3$ find the value of

$$(3x^2 - 2x + 3) - (4x^2 + 3x - 6) + (5x^2 - 7x + 2).$$

18. By how much does $(3a - b) - (5a + 4b)$ exceed $(-3a + 7b) + (-a - 3b)$?

19. What must be added to $(x - y) - (x + y)$ to produce 1?

The Multiplication of Signed Numbers

The Product of Signed Numbers

At the end of each day for three successive days the change in the price from the previous day of a certain stock was \$—2. What was the entire change for the three-day period?

The entire change in dollars was $+(-2) + (-2) + (-2)$
$= -2 - 2 - 2$
$= -6.$

Since the three addends are equal, the measure $+(-2) + (-2) + (-2)$ may be expressed as $(+3) \times (-2)$.

$\therefore (+3) \times (-2)$ is equivalent to -6.

In this product, the $+3$ is the multiplier and the -2 is the multiplicand. Hence, multiplying by a *positive* three, means taking the multiplicand as an *addend* three times.

Similarly, $(+5) \times (+3)$ means $+(+3) + (+3) + (+3) + (+3) + (+3)$
$= +3 + 3 + 3 + 3 + 3$
$= +15.$
$\therefore (+5) \times (+3) = +15.$

Also, $(+5) \times (-3)$ means $+(-3) + (-3) + (-3) + (-3) + (-3)$
$= -3 - 3 - 3 - 3 - 3$
$= -15.$
$\therefore (+5) \times (-3) = -15.$

When the multiplier is positive, the product is found by taking the multiplicand as an addend the required number of times. When the multiplier is negative, the product will be that found by taking the multiplicand as a subtrahend the required number of times.

Let us apply this to the product $(-3) \times (+5)$ in which we consider -3 the multiplier. Then,

the product $(-3) \times (+5)$ means $-(+5) - (+5) - (+5)$
$= -5 - 5 - 5$
$= -15.$

The result $(-3) \times (+5) = -15$ agrees with that obtained above when $+5$ was considered the multiplier in the same product.

Similarly, $(-3) \times (-5)$ means $-(-5) - (-5) - (-5)$
$= +5 + 5 + 5$
$= +15.$
$\therefore (-3) \times (-5) = +15.$

Therefore,
$$(+3) \times (+5) = +15.$$
$$(-3) \times (-5) = +15.$$
$$(+3) \times (-5) = -15.$$
$$(-3) \times (+5) = -15.$$

These examples illustrate that:

The product of two factors with the same sign is positive and the product of two factors with different signs is negative.

Exercise

A

1. Find the product of each of the following signed numbers:
(*a*) $+5$ and $+3$, (*b*) -5 and -8, (*c*) $+7$ and -9,
(*d*) -13 and $+7$, (*e*) $-\frac{1}{2}$ and $+\frac{3}{4}$, (*f*) $-\frac{7}{8}$ and $-\frac{4}{5}$.

2. State the square of
(*a*) -8, (*b*) $+6$, (*c*) (-7), (*d*) $+\frac{3}{4}$. (*e*) $-\frac{1}{2}$, (*f*) 0·3.

3. State the cube of
(*a*) -2, (*b*) $+3$, (*c*) -5, (*d*) 0·1, (*e*) $-0{\cdot}4$, (*f*) $-0{\cdot}2$.

B

4. Find the product of
(*a*) $+3\frac{3}{4}$ and $+1\frac{3}{5}$, (*b*) $-2{\cdot}54$ and $+7{\cdot}2$,
(*c*) $-3{\cdot}1416$ and $+10$, (*d*) $-0{\cdot}036$ and -100.

5. Find the continued product of each group of signed factors in the following:
(*a*) -2, -3, $+4$, $+2$ and $+1$; (*b*) -1, -5, -3 and $+4$;
(*c*) -1, $+2$ and $+7$; (*d*) -1, $+2$, -3, -4, $+5$ and $+2$;
(*e*) $+3$, $+2$, $+1$, -1 and -2; (*f*) -2, -3, -4 and $+3$.

Compare the sign of each product with those of its factors. State a rule for determining the sign of the product of a number of signed factors.

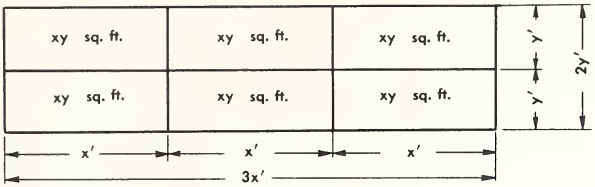
6. Evaluate each of the following powers:
(*a*) $(-1)^5$, (*b*) $(-1)^4$, (*c*) $(-2)^2$,
(*d*) $(-2)^3$, (*e*) $(-2)^4$, (*f*) $(-2)^5$,
(*g*) $(+3)^2$, (*h*) $(+3)^3$, (*i*) $(+3)^4$.

7. Calculate each of the following expressed products:
(*a*) $(-3)(-2)(-6)$, (*b*) $(-\frac{3}{4})(+\frac{1}{2})(-\frac{2}{3})$,
(*c*) $(+0{\cdot}03)(-0{\cdot}20)(+0{\cdot}35)$, (*d*) $(+1)^3$,
(*e*) $(-0{\cdot}2)(+0{\cdot}9)(-0{\cdot}4)(+10)^3$.

8. Find the value of each of the following expressions:
(*a*) $3(-2)^2$, (*b*) $-5(-2)^3$, (*c*) $-4(+3)^2$,
(*d*) $+3(-2)^3$, (*e*) $2(+3)^2 - 5(+3)$, (*f*) $5(-2)^3 - 4(-3)^3$.

Multiplication of Simple General Numbers

What is the area of a rectangle $3x$ feet long and $2y$ feet wide?



The area of the rectangle in square feet is $3x \times 2y$. A more concise way of expressing this product may be found from the diagram. Each small rectangle has an area of xy square feet. There are 6 such rectangles. The area of the large rectangle in square feet is $6xy$. Therefore, $3x \times 2y = 6xy$.

This product may be obtained by expressing $3x \times 2y$ as a product of its separate factors, thus, $3 \times x \times 2 \times y$. Since a product is not altered by changing the order of its factors

$$3x \times 2y = 3 \times 2 \times x \times y$$
$$= 6xy.$$

Similarly,
$$5x \times 2x = 5 \times 2 \times x \times x$$
$$= 10x^2.$$

Index Law for Multiplication of Powers of the Same Number

The **power of a number** is the product in which all the factors are equal. Thus, if the side of a square measures x inches, the measure of the area in square inches is $x \times x$ or x^2. The 2 indicates the number of times the base x is taken as a factor. This number is the **index** of the power. The product x^2 is the second power of x.

Similarly $x \times x \times x$ may be written x^3 and the product x^3 is the third power of x.

The product, $x^2 \times x^3 = x \times x \times x \times x \times x$
$$= x^5.$$

Again $a^4 \times a^2 = a \times a \times a \times a \times a \times a$
$$= a^6.$$

Similarly $x \times x^3 = x^4$, $a^5 \times a^2 = a^7$,
$b^3 \times b^2 \times b^4 = b^9$, $y^{10} \times y^{15} = y^{25}$,
$2^2 \times 2^3 = 2^5$, $(-10)^3 \times (-10)^2 \times (-10) = (-10)^6$.

When powers of the same base are multiplied, the product is a power of this same base and the index of this power is obtained by adding the indices of the separate factors.

The index law for multiplication is often stated briefly but not completely as: **To multiply powers of the same number add the indices.**

Example 1: Find the product of (*a*) $(-4ab)(+5c)$, (*b*) $(-5x^2y)(-3x^3y^2z)$.

Solution: (*a*) $(-4ab)(+5c)$ (*b*) $(-5x^2y)(-3x^3y^2z)$

$= -20abc.$ $= +15x^5y^3z.$

Example 2: Find the product of $-2x^2$, $+4xy$ and $-5x^3y$.

Solution: $(-2x^2)(+4xy)(-5x^3y)$

$= (-8x^3y)(-5x^3y)$

$= +40x^6y^2.$

Exercise

A

1. State the product of

(*a*) $6a$ and $5b$, (*b*) $5xy$ and $3ab$, (*c*) $3x$ and $4y$, (*d*) $4a$ and $3a$.

2. Multiply:

(*a*) $-3x \times +5y$, (*b*) $+4x \times +7x$, (*c*) $-6x \times -8y$,
(*d*) $-10ab \times -3a$, (*e*) $-4xy \times -4xy$, (*f*) $+4ab \times -3b$.

3. Simplify:

(*a*) $(x^2y)(xy^2)$, (*b*) $(2xy)(x^2y^2)$, (*c*) $x^4y^5 \times x^3y^2$,
(*d*) $(x^3)(y^2)$, (*e*) $3xy \times 5x^5y^4$, (*f*) $a^2b^4 \times a^5b$.

4. State in a more simple form:

(*a*) $x^2 \times x^3$, (*b*) $a^5 \times a^4$,
(*c*) $x \times x^5 \times x^3$, (*d*) $(-x^3)(-x^3)$,
(*e*) $x^2 \times y^2 \times y^3 \times y$, (*f*) $5x^2y \times 2xy$,
(*g*) $(-3a^2b)(-4ab)$, (*h*) $(-2x^3)(+5xy)$.

5. State the square of each of the following measures:

(*a*) $2x$, (*b*) $-3y$, (*c*) $-\frac{1}{2}x$,
(*d*) $5x^2$, (*e*) $-6xy$, (*f*) $-\frac{3}{4}x^2y$.

6. State the cube of:

(*a*) $3y$, (*b*) $-2x$, (*c*) $-0{\cdot}5x$,
(*d*) $-\frac{1}{2}x^2$, (*e*) $12x^2y$, (*f*) $0{\cdot}3ab^2$.

B

7. Express in simplified form each of the following products:

(*a*) $(5ab) \times (3bc)$, (*b*) $(-2a)(+5a)$,
(*c*) $(-7x)(-5y)$, (*d*) $(-3a)(+4b)$,
(*e*) $(-2x)^2$, (*f*) $(-3y)^3$,
(*g*) $(-x)^5$, (*h*) $(+x)^5$,
(*i*) $(-2x)(-2x)(-3x)$, (*j*) $(-\frac{3}{5}y)(+\frac{5}{8}y)(-\frac{4}{9}y)$.

8. Find the product of

(*a*) $-5ab^2$, $+6ab^3$; (*b*) $-6xy^3$, $-7x^3y^2$;
(*c*) $-2x^2$, $+4x^3$, $-3xy$; (*d*) $-4ab^4$, $-4a^3b$, $-3ab$;
(*e*) $-6x$, $-3y$, $-2x$, $+5y$; (*f*) x^5, x^4, $-x^3$, $-xy$.

9. Find the product of

(*a*) $1{\cdot}44ab^3$, $0{\cdot}12ab$; (*b*) $3{\cdot}14a^2 \times 0{\cdot}765nab$.

10. When $x = 3$, what is the numerical value of each of the following algebraic expressions:

(*a*) $2x$, (*b*) x^2, (*c*) $3x^2$,
(*d*) $(3x)^2$, (*e*) $(2x + 1)^2$, (*f*) $(3x - 10)^2$,
(*g*) $(1 - 2x)^3$, (*h*) $(3 - 5x)(2 + x)$, (*i*) $3x^2 + 2x + 5$?

11. Write in the simplest form:

(*a*) $x^2 + x^2$, (*b*) $x^2 \times x^2$,
(*c*) $x^2 - x^2$, (*d*) $2x + 5x$,
(*e*) $2x - 5$, (*f*) $2x \times 5x$,
(*g*) $-3x - 8x$, (*h*) $-3x \times -8x$,
(*i*) $-4x - (-3x)$, (*j*) $-4x + (-3x)$,
(*k*) $(-4x)(-3x)$, (*l*) $(-4x)^3(-3x)^2$.

12. Simplify:

(*a*) $+2(-7a)$, (*b*) $(-4)(+3a)$, (*c*) $-(+3)(+4x)$,
(*d*) $(-2a) - (-3a) - (4a)$, (*e*) $-(x - 3y) - (+4y)$,
(*f*) $(2x + 6y) + (3x - 7y) - (5x - y)$, (*g*) $0 - 3(a - b)$,
(*h*) $(-2t^2)^3$, (*i*) $(-2a^3b^4)^2$, (*j*) $(-2a)^3(-3a^2b)^2$.

C

13. If $p = x^2$, $q = 3x^2$ and $r = -4x^2$, find the value of

(*a*) $p + q + r$, (*b*) pqr, (*c*) $pq + qr$,
(*d*) $(p + q + r)^2$, (*e*) $p + q^2$, (*f*) $(p + q)^2$,
(*g*) $p^2 + q^2$, (*h*) $r^2 - q^2$, (*i*) $(r - q)^2$.

14. Express in suitable units the area of each rectangle the length and width of which are as follows:

(*a*) $3x$ in. and $5y$ in., (*b*) $5a$ ft. and $3a$ ft.,
(*c*) $\frac{3}{4}xy$ yd. and $\frac{5}{8}x$ yd., (*d*) $0{\cdot}35x$ mm. and $2{\cdot}5x$ cm.,
(*e*) $0{\cdot}875x$ km. and $25xy$ m., (*f*) $2{\cdot}5x$ ft. and $1{\cdot}5y$ in.

15. Express the volume in cubic units of a rectangular prism with length of base $0{\cdot}5x$ in., width of base $0{\cdot}25x$ in. and height $3x$ ft.

16. When $x = 3$ and $y = -5$, find the value of

(*a*) xy, (*b*) $3xy$, (*c*) $-4xy$,
(*d*) $-5x^2$, (*e*) $4x^2y^2$, (*f*) $2x^2 \times 3y$.

Division of Signed Numbers

Division of Signed Numbers

Examine these statements showing the quotients of signed numbers.

$$\text{Since } (+5) \times (+3) = +15, \text{ therefore } \frac{+15}{+3} = +5.$$

$$\text{Since } (+5) \times (-3) = -15, \text{ therefore } \frac{-15}{-3} = +5.$$

$$\text{Since } (-5) \times (-3) = +15, \text{ therefore } \frac{+15}{-3} = -5.$$

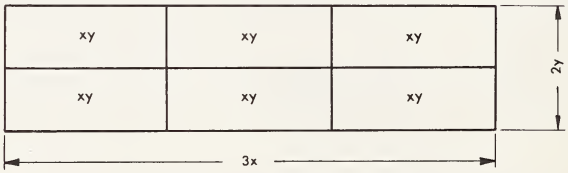
$$\text{Since } (-5) \times (+3) = -15, \text{ therefore } \frac{-15}{+3} = -5.$$

It will be observed that

The quotient of two numbers with like signs is positive and the quotient of two numbers with different signs is negative.

Division of Simple General Numbers

The measure of the area of a rectangle is represented by $6xy$ square inches. If the length is $3x$ inches, what is the width?



The width in inches is $\frac{6xy}{3x}$.

Three methods by which this quotient may be determined are illustrated below.

(1) By dividing the rectangle into its six equal rectangles each x inches long and y inches wide, it can be seen that the width in inches is $2y$.

$$\therefore \frac{6xy}{3x} = 2y.$$

(2) Let us find a number which multiplied by $3x$ gives a product of $6xy$. The number is $2y$.

$$\therefore \frac{6xy}{3x} = 2y.$$

(3) Consider $\frac{6xy}{3x}$ as a fraction. As in arithmetic, it may be reduced to lower terms by dividing the numerator and denominator by the same factor.

$$\therefore \frac{6xy}{3x} = \frac{2 \times 3 \times x \times y}{3 \times x}$$
$$= 2y.$$

Similarly, $\frac{12abc}{3ab} = 4c$, $\frac{15x}{5x} = 3$, and $\frac{3x}{3} = x$.

Example: Divide:

(*a*) $-8xy$ by $-2y$, (*b*) $+8ab$ by $-4b$, (*c*) $-6ac$ by $+3ac$.

Solution:

(*a*) $\frac{-8xy}{-2y} = +4x$. (*b*) $\frac{+8ab}{-4b} = -2a$.

(*c*) $-6ac \div +3ac = -2$.

Exercise

A

1. Express each of the following quotients in simplified form using signed numbers:

(*a*) $\frac{+6}{+2}$, (*b*) $\frac{-8}{-2}$, (*c*) $\frac{+12}{-6}$,

(*d*) $\frac{-10}{+5}$, (*e*) $\frac{-5}{+2}$, (*f*) $\frac{-7}{-3}$.

2. Find the quotient of each of the following divisions:

(*a*) $\frac{-10a}{+5a}$, (*b*) $\frac{+10a}{-10}$, (*c*) $\frac{-5b}{-b}$,

(*d*) $\frac{+24xy}{-\frac{1}{2}x}$, (*e*) $\frac{+2{\cdot}5ab}{-0{\cdot}5a}$, (*f*) $\frac{-ab}{-0{\cdot}25a}$,

(*g*) $\frac{+6abc}{-2ab}$, (*h*) $\frac{+9abc}{-3ac}$, (*i*) $\frac{5x}{x}$.

3. A rectangle has an area of $12ab$ square inches. Find the height which corresponds to each of the following bases:

(*a*) $2a$ inches, (*b*) $3b$ inches, (*c*) $4b$ inches,
(*d*) $6a$ inches, (*e*) $8a$ inches, (*f*) $9b$ inches.

4. A prism has a volume of $24abc$ cubic inches. What will be the measure of the area of its base for each of the following heights?

(*a*) $3a$ inches, (*b*) $8b$ inches, (*c*) $12c$ inches,
(*d*) $12a$ inches, (*e*) $\frac{1}{2}a$ inches, (*f*) $0{\cdot}08c$ inches.

B

5. Divide:

(*a*) $-\frac{3}{4}$ by $+\frac{1}{8}$, (*b*) $+2\frac{1}{2}$ by $-\frac{3}{5}$,
(*c*) $+21$ by -24, (*d*) -15 by $-\frac{3}{8}$,
(*e*) $-\frac{3}{5}$ by $+6$, (*f*) $+2\frac{1}{3}$ by $-1\frac{3}{4}$.

6. Evaluate:

(*a*) $\dfrac{+8}{+0{\cdot}02}$, (*b*) $\dfrac{-0{\cdot}02}{-8}$, (*c*) $\dfrac{+0{\cdot}1}{-0{\cdot}01}$,

(*d*) $\dfrac{-0{\cdot}5}{+0{\cdot}04}$, (*e*) $\dfrac{-34{\cdot}3}{+0{\cdot}7}$, (*f*) $\dfrac{-12{\cdot}96}{-0{\cdot}036}$.

7. Simplify:

(*a*) $4ab \div \frac{1}{3}a$, (*b*) $(-8\frac{1}{4}xy) \div 4$,
(*c*) $(+2\frac{1}{4}x) \div (-1\frac{1}{3}x)$, (*d*) $(-\frac{3}{4}xyz) \div \frac{5}{6}xy$,
(*e*) $(-\frac{3}{4}xy) \div (+6x)$, (*f*) $(-3\frac{1}{3}yz) \div (-2\frac{1}{2}z)$.

8. If $x = 3$, $y = 4$, $a = 2$ and $b = 1$, find the value of

(*a*) $\dfrac{2x}{3y}$, (*b*) $\dfrac{-5xa}{6b}$, (*c*) $\dfrac{+4xb}{-9ay}$,

(*d*) $\dfrac{-2x^2y}{-9a}$, (*e*) $\dfrac{3x+4y}{5a-12b}$, (*f*) $\dfrac{6ax-2by}{3bx+6b}$.

9. A triangle has an area of $15xy$ square inches. What will be the measure of its height corresponding to each of the following bases:

(*a*) $2x$ inches, (*b*) $3y$ inches, (*c*) $6x$ inches,
(*d*) $10x$ inches, (*e*) $20y$ inches, (*f*) $30y$ inches?

Index Law for the Division of Powers of the Same Number

The quotient when a^5 is divided by a^3 may be expressed in fractional form as $\dfrac{a^5}{a^3}$. This fraction may be reduced to lower terms by dividing the numerator and denominator by the same factor.

$$\frac{a^5}{a^3} = \frac{a \times a \times a \times a \times a}{a \times a \times a}$$
$$= a \times a$$
$$= a^2.$$

This same result may be obtained by applying the principle that the operation of division is opposite to that of multiplication. To simplify the quotient $\dfrac{a^5}{a^3}$, we might ask, "What power of a multiplied by a^3 will give a^5?" $a^2 \times a^3$ gives a^5, therefore $\dfrac{a^5}{a^3} = a^2$. Also, $x^{10} \div x^2 = x^8$ and $15^6 \div 15^4 = 15^2$.

When powers of the same base are divided, the quotient is a power of this same base and the index of this power is obtained by subtracting the index of the divisor from that of the dividend.

The index law for division is often stated briefly but not completely—**To divide powers of the same number subtract the indices.**

Example: Simplify $+15a^5 \div -3a^2$.

Solution: $\dfrac{15a^5}{-3a^2} = -5a^3$.

Exercise

A

1. State the quotient in each of the following:

(*a*) $\dfrac{x^5}{x^2}$, (*b*) $\dfrac{x^4}{x^2}$, (*c*) $\dfrac{x^6}{x^3}$, (*d*) $\dfrac{a^3}{a^2}$,

(*e*) $\dfrac{a^5b^3}{ab^2}$, (*f*) $\dfrac{m^6n^5}{m^3n^4}$, (*g*) $\dfrac{nx^3}{nx}$, (*h*) $\dfrac{k^3m^4n^6}{km^2n^3}$.

2. State the quotient in each of the following:

(*a*) $\dfrac{10^7}{10^4}$, (*b*) $\dfrac{10^3}{10}$, (*c*) $\dfrac{2^{10}}{2^5}$, (*d*) $\dfrac{3^6}{3^2}$.

3. Divide:

(*a*) $\dfrac{15x^7}{5x^3}$, (*b*) $\dfrac{-18a^4}{-3a^3}$, (*c*) $\dfrac{-20a^3b}{5ab}$, (*d*) $\dfrac{-12x^4y^3}{-6xy^3}$,

(*e*) $\dfrac{+8x^2}{-4x}$, (*f*) $\dfrac{-4x^4}{x^3}$, (*g*) $\dfrac{-12ab^2}{+3ab}$, (*h*) $\dfrac{-15a^3b^2c^2}{-5a^2c}$.

B

4. Divide:

(*a*) $\dfrac{36abc}{9c}$, (*b*) $\dfrac{48a^4b^5}{-8a^2b^3}$, (*c*) $\dfrac{-18ab^3}{-3b}$,

(*d*) $\dfrac{1{\cdot}44b^3}{-3{\cdot}6b}$, (*e*) $\dfrac{4{\cdot}2a^6b^5}{0{\cdot}06ab^5}$, (*f*) $\dfrac{\frac{1}{2}a^2b^2}{\frac{1}{4}ab}$.

5. Simplify:

(*a*) $4a \times \dfrac{a}{2}$, (*b*) $\dfrac{8d}{6}$, (*c*) $\dfrac{10y^2}{5y^2}$,

(*d*) $4a \div \dfrac{a}{2}$, (*e*) $6r^3 \div \frac{1}{4}r$, (*f*) $(6y)^3 \div (2y)^2$.

6. Simplify:

(*a*) $\dfrac{a^4 \times a^5}{a^3}$, (*b*) $\dfrac{b \times b^2 \times b^4}{b^7}$, (*c*) $\dfrac{10^5 \times 10^4}{10^6}$,

(*d*) $\dfrac{10 \times 10^6}{10^3}$, (*e*) $a^6 \div a^4 + a^2$, (*f*) $\dfrac{-8a^4 \times -6a^3}{-4a \times -3a^2}$,

(*g*) $\dfrac{\pi r^2}{r}$, (*h*) $\dfrac{\pi r^2}{\pi}$, (*i*) $\dfrac{\frac{1}{3}\pi r^2 h}{\pi r}$.

7. (*a*) How many squares each measuring x inches to a side will be equivalent in area to a rectangle $5x$ inches long and $3x$ inches wide?
(*b*) How many cubes x inches to a side could be fitted into a rectangular box the inside dimensions of which are $8x$ inches, $6x$ inches and $5x$ inches?

8. How many times is the area of a rectangle, of which the length is $3a$ and the width is $2b$, contained in a rectangle of which the length is $8a$ and the width is $12b$?

9. The volume of a triangular prism is $12a^2b$ cubic inches. If its height is $3a$ inches what is the area of its base?

10. The volume of a rectangular prism in cubic feet is $15x^2y$. If the lengths of the sides of the base are $3x$ feet and y feet respectively, what is its height?

11. If the circumference of a circle is $154x$ inches, find its diameter. (Use $\pi = \frac{22}{7}$)

Substitution with Negative Numbers

Some measurements such as the length of a room or the circumference of a circle rarely, if ever, involve negative numbers. On the other hand, many measurements such as those of temperature and voltage do require negative numbers.

The relationship between the measure of a temperature in Centigrade readings and the measure of the same temperature in Fahrenheit readings is given by the formula $C = \frac{5}{9}(F - 32)$, in which F represents the measure of the temperature in Fahrenheit degrees and C represents the measure of the temperature in Centigrade degrees.

This formula may be used to find the Centigrade reading of a temperature when the Fahrenheit reading is known.

Example 1: During the winter of 1959 the temperature at a northern meteorological station reached a low of $-28°F$. What would this reading be on a Centigrade scale?

Solution: By substituting $F = -28$ in the formula the temperature in degrees Centigrade was $\frac{5}{9}(-28 - 32)$

$$= \tfrac{5}{9}(-60)$$
$$= -33\tfrac{1}{3}.$$

Example 2: Evaluate $5x^2 - 3x + 2$ when $x = -2$.

Solution:
$$\begin{aligned} & 5x^2 - 3x + 2 \\ &= 5(-2)^2 - 3(-2) + 2 \\ &= 5 \times 4 - (-6) + 2 \\ &= 20 + 6 + 2 \\ &= 28. \end{aligned}$$

Exercise

A

1. If $x = -5$, what is the value of (*a*) $3x$, (*b*) x^2, (*c*) x^3?

2. If $a = -12$, what is the value of

(*a*) $3a$, (*b*) $\frac{1}{2}a$, (*c*) $\frac{a}{3}$,

(*d*) $\frac{12}{a}$, (*e*) $\frac{36}{2a}$, (*f*) $\frac{9}{a}$?

3. If $y = -2$, what is the value of (*a*) $3y$, (*b*) $4y^2$, (*c*) $3y^3$?

B

4. When $x = -3$, what is the numerical value of each of the following expressions:

(*a*) $2x$, (*b*) x^3, (*c*) $3x^2$, (*d*) $4 - x$,
(*e*) $(x - 3)^2$, (*f*) $(1 - x)^2$, (*g*) $(2x - 3)^2$?

5. If $b = -4$, find the value of

(*a*) $6 - b$, (*b*) $1 - b$, (*c*) $8 + b$, (*d*) $7 - 3b$, (*e*) $b - 4$.

6. If $x = -2$, find the value of

(*a*) $(3x)^2$, (*b*) $(2x + 1)^2$, (*c*) $(3x - 10)^2$,
(*d*) $(1 - 2x)^2$, (*e*) $(3 - 5x)(2 + x)$, (*f*) $3x^2 + 2x + 5$.

7. Find the values of $2x + 5$ as x is given each integral value from -6 to $+6$. Arrange the values in a table.

8. Find the value of $16 - 3x$ as x varies from $x = -3$ to $x = +6$. Use integral values only for x.

9. If $n = x^2 - 2x + 1$, find the value of n for the following values of x: $x = -2$, $x = -1$, $x = 0$, $x = +1$, $x = +2$, $x = +3$.

10. What is the numerical value of $2x^2 - 5x + 2$ for each of the following values of x:

(*a*) $x = -2$, (*b*) $x = -1$, (*c*) $x = 0$,
(*d*) $x = \frac{1}{4}$, (*e*) $x = \frac{1}{2}$, (*f*) $x = 1$,
(*g*) $x = 1\frac{1}{2}$, (*h*) $x = 2$, (*i*) $x = 3$?

11. When $x = -2$ and $y = -3$, find the numerical value of each of the following expressions:

(*a*) $x + y$, (*b*) $x - y$, (*c*) $x^2 + y^2$,
(*d*) $x^2 - y^2$, (*e*) $x^3 + y^3$, (*f*) $x^3 - y^3$.

12. The formula $F = \frac{9}{5}C + 32$ is used to change readings of temperature which have been obtained on a Centrigrade scale to a corresponding Fahrenheit reading. Find the temperature in degrees Fahrenheit corresponding to each of the following readings of a thermometer in degrees Centigrade which is located in a refrigeration plant:

(*a*) $+7°$, (*b*) $-5°$, (*c*) $-15°$, (*d*) $-40°$.

13. Using the relationship $C = \frac{5}{9}(F - 32)$, change the following temperature readings in degrees Fahrenheit to readings in degrees Centigrade:

(*a*) 68°, (*b*) 32°, (*c*) 0°, (*d*) $-13°$, (*e*) $-40°$.

C

14. (*a*) When $x = 2$ and $y = 3$ what is the value of $x + y$ and $x^2 + 2xy + y^2$?
(*b*) Find the values of $x + y$ and $x^2 + 2xy + y^2$ corresponding to the sets of values of x and y in the table. Complete the table.

x	$+2$	$+2$	-2	-2	$+5$	$+\frac{1}{2}$
y	$+3$	-3	$+3$	-3	-2	$+\frac{3}{4}$
$x + y$						
$x^2 + 2xy + y^2$						

Do your results indicate any relationship between the values of $x^2 + 2xy + y^2$ and $x + y$?

15. Make the necessary calculations to complete the following table.

x	$+5$	$+6$	-3	-3	$+\frac{1}{2}$	$+0{\cdot}5$
y	$+4$	$+2$	$+2$	-2	$+\frac{1}{4}$	$+0{\cdot}2$
$x + y$						
$x - y$						
$x^2 - y^2$						

Do your results indicate any relationship between $x + y$, $x - y$ and $x^2 - y^2$?

Review Exercise

A

1. State in the simplest possible form:

(*a*) $a + a + a + a$, (*b*) $a \times a \times a$,
(*c*) $ab + ba$, (*d*) $3x - 5y + 4y - 3x$,
(*e*) $x^2 + x^2 + x^2$, (*f*) $x^2 \times x^2 \times x^2$.

2. Complete these statements:

(*a*) Three times a number diminished by 5 is .....
(*b*) A number x increased by 10 is .....
(*c*) The product of 7 and x is .....
(*d*) The sum of 7 and x is .....
(*e*) The quotient of 7 by x is .....

3. In each of the following formulas state which numbers are variables and which are constants:

(*a*) $A = \frac{1}{2}bh$, (*b*) $A = \pi r^2$, (*c*) $C = \frac{5}{9}(F - 32)$,
(*d*) $V = \pi r^2 h$, (*e*) $A = bh$, (*f*) $C = 2\pi r$.

4. Find the product of:

(*a*) $(-6a)(-3a)$, (*b*) $5x^2 \times (-6x)$, (*c*) $(-4)(-3)(-2)$,
(*d*) $a^4 \times a^3$, (*e*) $(+4xyz)(+8xyz)$, (*f*) $\frac{1}{2}x^2 \times \frac{1}{4}x^2$,

5. Find the quotients in the following divisions:

(*a*) $\frac{16x}{8x}$, (*b*) $\frac{16x}{8}$, (*c*) $\frac{16x}{x}$,

(*d*) $\frac{-16x}{-4}$, (*e*) $\frac{-5xy}{-x}$, (*f*) $\frac{4x^2}{4x}$.

B

6. Evaluate $4x^2 + 4xy + y^2$ when $x = 5$ and $y = -3$.

7. Simplify the following expressions:

(*a*) $6x + 5y - 3z + 4x - 8y - z$,
(*b*) $2x^2 - 3x + 5 - 7x^2 - 5x - 10$,
(*c*) $2ab - 3ba + 3ac - 6ca$,
(*d*) $2x + 3y - 5x - 5y - 8x + 7y$,
(*e*) $2x^3 - 3x^3 - 4x^3 + 8x^2$.

8. Remove the brackets and simplify:

(*a*) $(x + z) - (y - z)$, (*b*) $(2a - b) - (b - 3a)$,
(*c*) $(2a + 3b) + (3a - 5b)$, (*d*) $-(2x - 6) + (5x - 8)$,
(*e*) $(6 - x) + (9 - x) - (3 - x)$,
(*f*) $(-4) + (-x) + (-5) - (+x)$.

9. What is the excess of $8x - 7y$ over $5x - 10y$?

10. What is the volume of a rectangular tank which is $5x$ inches long, $7x$ inches wide and $\frac{1}{3}x$ inches deep?

11. Simplify:

(a) $(2x)^3$, (b) $(5x)^2$, (c) $(-3x)^2$,

(d) $(-3x)^3$, (e) $(2x)^2(-2x)^2$, (f) $(-1)^{10}$,

(g) $\frac{7^2}{7^2}$, (h) $\frac{x^2}{x^2}$, (i) $\frac{-18x^4y^5}{-2xy^3}$,

(j) $\frac{1{\cdot}44x}{-1{\cdot}2x}$, (k) $\frac{2 \times 15xy}{3y}$, (l) $\frac{-6x \times -3y}{-9xy}$,

(m) $\frac{7x^2 \times (-5x^2)}{+0{\cdot}1x}$, (n) $\frac{(0{\cdot}1)^3}{0{\cdot}1}$, (o) $\frac{10^6}{10^2}$.

12. If $a = 0{\cdot}05$ and $b = 0{\cdot}04$, find the value of

(a) $a + b$, (b) ab, (c) $\frac{a}{b}$,

(d) $\frac{b}{a}$, (e) a^2, (f) b^3,

(g) $a + a^2$, (h) $(a + 1)^2$, (i) $2a - 5b$,

(j) $(a - b)^2$, (k) $(b - a)^2$, (l) $100a - 25b$.

13. If $a = \frac{1}{16}$ and $b = \frac{5}{8}$, find the value of

(a) $a + b$, (b) $a - b$, (c) ab, (d) $\frac{a}{b}$.

14. Find the area of a triangle with a base of $8x''$ and an altitude to this base of $10x''$.

15. Compare the value of $x^2 + 5x + 6$ with $(x + 2)(x + 3)$ for at least two positive values of x and at least two negative values of x.

16. Determine if $x^2 + 16$ has the same value as $(x + 4)^2$ when $x = 5$.

17. What is the value of $5x^2 - 4x + 7$ when

(a) $x = 4$, (b) $x = 3$, (c) $x = 2$,

(d) $x = 1$, (e) $x = 0{\cdot}1$, (f) $x = 0$,

(g) $x = -2$, (h) $x = -3$, (i) $x = -4$?

C

18. The volume of a cylinder is given by the formula $V = \frac{\pi d^2 h}{4}$ where d and h are the measures of the diameter and height in the same linear units respectively. Complete the following table for various cylinders as indicated.

Diameter in inches		1	1	1	1	2	3	4	5	6	7
Height in inches		1	2	3	4	1	1	1	1	1	1
Volume in cubic inches	Expressed										
	Simplified										

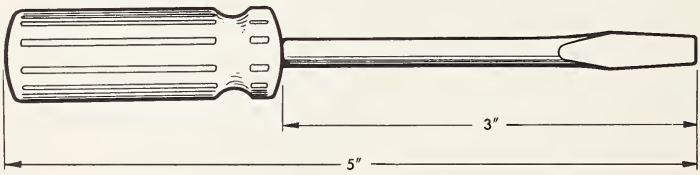
CHAPTER 12

Equations

Algebraic methods provide us with solutions for many problems for which arithmetical methods are not suitable. Since the methods used are similar for both simple and difficult problems, we shall begin with simple problems.

Consider the following problem:

The shank of a screwdriver is 3″ long; the over-all length is 5″. What is the length of the handle?



In this situation there are three related measures. They are the measures of (1) the length of the shank, (2) the total length of the screwdriver and (3) the length of the handle.

In the algebraic method of solving problems, the unknown measures are represented by letters. In this problem the measure of the length of the handle is the *unknown*.

Therefore, let the length of the handle in inches be x.

The length of the shank in inches is 3.

The total length in inches is $x + 3$.

But the total length in inches is given as 5.

Here we have the measure of the same quantity, in the same units, expressed in two different ways.

$$\therefore x + 3 = 5.$$

This is a statement expressing the equality of two measures and is called an **algebraic equation.**

The algebraic method always involves an equation. To complete the solution it will be necessary to find the value of the letter which will make the

statement of the equation true. This operation is called **solving the equation.** The correct value of the letter is called the **root of the equation.**

By inspection, we can see that $x = 2$ in this equation. Therefore, the length of the handle is 2 inches.

Solution of Simple Equations

The solutions of a great variety of equations are dependent upon one or more of four simple principles.

Let us consider each of the following situations:

(1) John had two boxes, A and B, each containing the same number of pencils. Then John added two pencils to each box. After this was done, how did the number of pencils in A compare with the number of pencils in B? You will agree that the number in A now is equal to the number in B. Other similar situations may be described to illustrate the principle that,

If equal measures are added to equal measures the sums are equal.

(2) In a similar way we could illustrate that,

If equal measures are subtracted from equal measures the differences are equal.

(3) Let us again begin with John's two boxes, A and B, each containing an equal number of pencils. John then doubles the number of pencils in each box. How will the number in A now compare with the number in B? Again the number in A equals the number in B and we see that,

If equal measures are multiplied by equal measures the products are equal.

(4) Similarly,

If equal measures are divided by equal measures the quotients are equal.

These principles are four of the **axioms** which play an important role in mathematics.

Let us see how the application of these principles will help us to solve equations for unknown measures.

Example 1: Solve $x - 5 = 10$.

The measure $x - 5$ becomes x if 5 is added to it. If we add 5 to the left side of an equation we must add 5 to the right side to keep the sums equal. The solution might be written as:

$$x - 5 = 10.$$

Add 5 to the measure of each side $\quad x - 5 + 5 = 10 + 5.$

$$\therefore x = 15.$$

Example 2: Solve $x + 8 = 13$.

Solution:

$$x + 8 = 13.$$

Subtract 8 from each side $\quad x + 8 - 8 = 13 - 8.$

$$\therefore x = 5.$$

Example 3: Solve $\frac{1}{3}x = 9$.

Solution:

$$\frac{1}{3}x = 9.$$

Multiply each side by 3

$$3 \times \frac{1}{3}x = 3 \times 9.$$

$$\therefore x = 27.$$

Example 4: Solve $3x = 15$.

Solution:

$$3x = 15.$$

Divide each side by 3

$$\frac{3x}{3} = \frac{15}{3}.$$

$$\therefore x = 5.$$

Exercise

A

State in words the complete solution for each of the equations in this exercise.

1. $3x = 27$.
2. $5x = 35$.
3. $4y = 28$.
4. $5z = 200$.
5. $8y = 128$.
6. $3x = 1{\cdot}2$.
7. $4x = 4$.
8. $8 = 2x$.
9. $66 = 6k$.
10. $10x = 10$.
11. $5x = 2$.
12. $6 = 18a$.
13. $2 = 6y$.
14. $8k = 4$.
15. $2x = 0$.
16. $\frac{1}{2}x = 6$.
17. $\frac{1}{3}y = 5$.
18. $\frac{1}{4}m = 8$.
19. $\frac{1}{8}y = \frac{1}{2}$.
20. $3 = \frac{1}{2}x$.
21. $\frac{n}{3} = 7$.
22. $\frac{x}{7} = 2$.
23. $10 = \frac{b}{8}$.
24. $\frac{n}{8} = \frac{5}{8}$.
25. $\frac{n}{10} = \frac{2}{10}$.
26. $x + 4 = 7$.
27. $x + 3 = 11$.
28. $x + 4 = 4$.
29. $10 = x + 2$.
30. $13 = 11 + y$.
31. $25 = n + 16$.
32. $x + 0{\cdot}1 = 0{\cdot}5$.
33. $a + 1{\cdot}2 = 1{\cdot}8$.
34. $b + 2 = 2\frac{1}{2}$.
35. $x + \frac{1}{4} = \frac{1}{2}$.
36. $a + \frac{1}{18} = \frac{1}{2}$.
37. $1 = \frac{1}{8} + a$.
38. $a - 3 = 4$.
39. $c - 5 = 1$.
40. $x - 5 = 5$.
41. $20 = a - 15$.
42. $2 = a - 6$.
43. $3 = a - 3$.
44. $x - 2{\cdot}5 = 5$.
45. $b - 1{\cdot}2 = 1{\cdot}4$.
46. $y - 7 = 0$.
47. $2x + 1 = 3$.
48. $3x + 5 = 11$.
49. $2x - 1 = 5$.

Solution of Equations

Before considering solutions of problems in their entirety, we shall discuss more fully the process of solving equations. The following examples will

illustrate how equations can be solved by the application of one or more of the four axioms.

Example 1: Solve $5x - 7 = 3x - 2$.

We noticed that an equation having an *unknown term* on one side and a *known term* on the other side could be solved directly by applying axiom 4. Hence in this equation we shall endeavour to remove the term -7 from the left side and the term $+3x$ from the right side.

Solution: $5x - 7 = 3x - 2$.

Add 7 to each side $5x = 3x + 5$.

Subtract $3x$ from each side $2x = 5$.

Divide each side by 2 $x = 2{\cdot}5$.

Verification: When $x = 2{\cdot}5$, then $5x - 7$
$= 5 \times 2{\cdot}5 - 7$
$= 12{\cdot}5 - 7$
$= 5{\cdot}5$.

When $x = 2{\cdot}5$, then $3x - 2$
$= 3 \times 2{\cdot}5 - 2$
$= 7{\cdot}5 - 2$
$= 5{\cdot}5$.

Since $5x - 7$ and $3x - 2$ have the same value when $x = 2{\cdot}5$, therefore $x = 2{\cdot}5$ is the correct solution.

Example 2: Solve $2x + 5 = 4x + 8$.

Solution: $2x + 5 = 4x + 8$.

Subtract 5 from each side.

$2x = 4x + 3$.

Subtract $4x$ from each side.

$-2x = 3$.

Divide each side by -2.

$x = -1{\cdot}5$.

or

Solution: $2x + 5 = 4x + 8$.

Subtract $2x$ from each side.

$5 = 2x + 8$.

Reverse the equation.

$2x + 8 = 5$.

Subtract 8 from each side.

$2x = -3$.

Divide each side by 2.

$x = -1{\cdot}5$.

Verification:

When $x = -1{\cdot}5$, then $2x + 5$
$= 2(-1{\cdot}5) + 5$
$= -3{\cdot}0 + 5$
$= 2$.

When $x = -1{\cdot}5$, then $4x + 8$
$= 4(-1{\cdot}5) + 8$
$= -6 + 8$
$= 2$.

Since $4x + 8$ and $2x + 5$ have the same value when $x = -1{\cdot}5$, therefore $x = -1{\cdot}5$ is the correct solution.

Example 3: Solve $5y + 3 - 2y = 8 + 6y - 5$.

Solution: $5y + 3 - 2y = 8 + 6y - 5$.

Simplify each measure $3y + 3 = 6y + 3$.

Subtract $6y$ from each side $-3y + 3 = 3$.

Subtract 3 from each side $-3y = 0$.

Divide each side by -3 $+y = 0$.

Verification:

When $y = 0$, $5y + 3 - 2y = 0 + 3 - 0 = 3$ and $8 + 6y - 5 = 8 + 0 - 5 = 3$.

Since $5y + 3 - 2y$ and $8 + 6y - 5$ have the same value when $y = 0$, therefore $y = 0$ is the correct solution.

Exercise B

Solve the following equations and in each case verify your solution.

1. $2r + 3 = 7$.

2. $5x = 3x + 8$.

3. $72n + 200 = 272$.

4. $66k = 10 + 6k$.

5. $8a + 5 = 5$.

6. $4x - 11 = 3x$.

7. $0{\cdot}04x - 8 = 20$.

8. $3x + 5 = 2x + 8$.

9. $5m - 8 = 16 + m$.

10. $3x + 4x = 49$.

11. $8y - 7 = 3y + 23$.

12. $4x + 42 = 11x$.

13. $5 - 2w = 1$.

14. $\frac{1}{3}m - 7 = 2$.

15. $\frac{2}{3}s = 8$.

16. $5 = 4 - 3x$.

17. $3{\cdot}14d = 15{\cdot}7$.

18. $6x - 2 = 3x + 10$.

19. $5x + 7 = 8x + 13$.

20. $0{\cdot}5x - 0{\cdot}04 = 0{\cdot}2x + 0{\cdot}32$.

21. $8x + 5 = 3x - 10$.

22. $6 - 10a = 3 - 16a$.

23. $16x - 12x - 8 = 10 - 4x$.

24. $(3x + 2) + (5x - 7) = 35$.

25. $(7x - 6) - (3x + 4) = 10x - 46$.

26. $(5x + 3) - (2x - 7) = 16$.

27. $5 + (7x - 2) = (3x + 4) - (2x + 3)$.

28. $\frac{2}{3}x + 7 = 11$.

29. $\frac{3}{4}x = 9$.

30. $\frac{5}{8}x = 6\frac{1}{4}$.

31. $\frac{5}{8}x - 5 = 1 - \frac{1}{8}x$.

32. $7y - (3y + 4) = 16$.

33. $11x - (3x + 24) = 4x$.

34. Is $x = 5$ the correct solution for $x^2 - x = 20$?

35. If $x = 7$, find the value of y if $3x + 5y = 31$.

36. In the equation $3y = 2x - 7$ we know that $y = 9$, find the value of x.

37. In the equation $y - 8x = 2$, we know $y = -3$, find the value of x.

38. Is $x = 3$ the correct solution for $4x + 8 = x + 17$?

39. Is $x = 4$ a correct solution for the equation $x^2 + 6x = 40$?

40. Which of the numbers 2, 3, $\frac{1}{2}$ or $\frac{1}{3}$ is the correct value of x to satisfy the equation $16x - 3 = 10x - 1$?

41. In the equation $3a + 5b = 25$, we know that $b = 2$. What is the value of a?

42. If x has the value $2\frac{1}{4}$ in the equation $x + y = 10$, what is the value of y?

43. If in the equation $y = ma + b$, $m = \frac{1}{2}$, $a = 16$ and $b = 6$ what is the value of y?

44. Find the value of R in the electrical formula $IR = E$ if $I = 0{\cdot}3$ and $E = 60$.

45. Is $x = 9$ a solution for $8x + 7 = 12x - 28$?

Formulas

When a formula is used, it is often necessary to express the value of one letter in terms of the others. The method of solving for one of the letters is the same as that used when the coefficients are numerical.

Example 1: In the formula $v = a + ut$, find the numerical value of a when $v = 15, u = 2$ and $t = 3$.

There are two methods of attack.

Solution 1: Substitute the values $v = 15$, $u = 2$ and $t = 3$ in the formula $v = a + ut$ immediately.

$$15 = a + 2 \times 3.$$

Reverse the equation $a + 6 = 15.$

Subtract 6 from each side $a = 9.$

Solution 2: $v = a + ut.$

Reverse the equation $a + ut = v.$

Subtract ut from each side $a = v - ut.$

Substitute $v = 15$, $u = 2$ and $t = 3$.

$$\begin{aligned} \therefore a &= 15 - 2 \times 3 \\ &= 15 - 6 \\ &= 9. \end{aligned}$$

Example 2: In the formula $d = vt$, solve for v.

Solution:

$$d = vt.$$

Reverse the equation. $vt = d.$

Divide each side by t. $v = \frac{d}{t}.$

Exercise

A

1. If $A = bh$, state the value of b if $A = 15$ and $h = 5$.
2. $p = a + b + c$ when $p = 18$, $a = 6$ and $c = 5$, find b.
3. If $a + b + c = 180$ and $a = 90$, $c = 30$, find b.
4. In the formula $E = IR$, find I when $E = 110$ and $R = 22$.
5. In the formula $d = rt$, $d = 400$, $r = 50$, find t.
6. When $A = bh$, solve for b.
7. When $c = \pi d$, solve for d.
8. When $c = 2\pi r$, solve for r.
9. When $p = a + b + c$, solve for c.

B

10. If $A = bh$, find h if $A = 12{\cdot}96$ when $b = 7{\cdot}2$.
11. If $p = a + b + c$, find b if $p = 356$ when $a = 87$ and $c = 59$.
12. If $A = \frac{1}{2}bh$, find h if $A = 24$ when $b = 18$.
13. If $A = \pi dh$, find h when $\pi = \frac{22}{7}$, $d = 28$ and $A = 616$.
14. If $V = \frac{1}{3}Ah$, find A if $V = 249$ when $h = 18$.
15. If $p = 2x + 2y$, find x when $p = 84$ and $y = 26$.
16. If $A = \frac{1}{2}hx + \frac{1}{2}hy$, find y if $A = 360$ when $h = 18$, $x = 10$.
17. When $A = \pi dh$, solve for d.
18. When $E = IR$, solve for I.
19. When $a + b + c = p$, solve for a.
20. When $d = vt$, solve for t,
21. When $I = Prt$, solve for r.
22. When $ax = b$, solve for x.
23. When $my = b + d$, solve for y.
24. When $am + b = c$, solve for b.
25. When $mz + b = k$, solve for z.
26. When $\dfrac{x}{a} = c$, solve for x.
27. When $\dfrac{x}{m} = \dfrac{n}{k}$, solve for x.
28. When $\dfrac{m}{n} = \dfrac{x}{a}$, solve for x.

C

29. When $5x - b = 2x + d$, solve for x.
30. When $90 - A = C$, solve for A.

31. When $180 - a - b = c$, solve for a.

32. When $\frac{1}{2}h(x + y) = A$, solve for x.

33. When $p = a - (b + c)$, solve for b.

34. When $b = \frac{1}{12}awn$, solve for a.

35. When $k = m + (n - 1)c$, solve for c.

36. When $v = at + 16t^2$, solve for a.

Problems

The examples given below contain in simple form many of the essential processes used in the solution of all mathematical problems. The algebraic method of solving problems is the same as the method used to develop formulas in the earlier part of this course. Simple problems are used here in order that the method may be emphasized.

Example 1: If 45 is added to a certain number, the sum is the same as four times the number. Find the number.

Solution:

Let the number be x.

Then the number increased by 45 is $x + 45$.

Four times the number is $4x$.

$$\therefore 4x = x + 45$$
$$4x - x = 45$$
$$3x = 45$$
$$x = 15$$

The number is 15.

Check: $15 + 45$ make 60.

4×15 is 60.

Therefore, the solution is correct.

Example 2: A strip of sheet metal 21″ long is to be marked at 3 places so that it may be bent to form a triangle, two sides of which are respectively 3″ and 5″ longer than the third side. Find the length of each section.

Solution:

Let the length of the shortest section in inches be x.

Then the length of the middle section in inches is $x + 3$.

Then the length of the longest section in inches is $x + 5$.

Total length of the three sections in inches is $x + (x + 3) + (x + 5)$

$$= x + x + 3 + x + 5$$
$$= 3x + 8.$$

But the total length of the sections in inches is given as 21.

$$\therefore 3x + 8 = 21$$
$$3x = 13$$
$$x = 4\tfrac{1}{3}.$$

Therefore, the length of the shortest section is $4\frac{1}{3}$ inches.

The length of the middle section is $7\frac{1}{3}$ inches.

The length of the longest section is $9\frac{1}{3}$ inches.

Check: Each condition in the problem is checked independently.

(*a*) $7\frac{1}{3}''$ is 3″ longer than the shortest section.

(*b*) $9\frac{1}{3}''$ is 5″ longer than the shortest section.

(*c*) Total length of the section in inches is $4\frac{1}{3} + 7\frac{1}{3} + 9\frac{1}{3}$
$= 21.$

Therefore, the solution above is correct.

Example 3: A rectangle, which is twice as long as it is wide, has a perimeter of 144 feet. Find its length and width.

Solution: Let the width of the rectangle in feet be x.

Then the length of the rectangle in feet is $2x$.

Then the perimeter of the rectangle in feet is $2(x + 2x)$
$= 6x.$

But the perimeter of the rectangle in feet is given as 144.

$$\therefore 6x = 144.$$
$$x = 24.$$

The width of the rectangle is 24 ft. The length of the rectangle is 48 ft.

Check: The length is twice the width.

The perimeter in feet is $2(24 + 48)$
$= 2 \times 72$
$= 144.$

Therefore the solution is correct.

Suggestions for Solving Problems

1. List the quantities involved in the situation with their respective units of measurement and, as far as possible, their arithmetical measures.

2. Express, algebraically, the unknown measures.

3. Express in two different ways the measure for the same or corresponding quantities.

4. State the equality of the two expressions as an equation.

5. Solve the equation.

6. State the conclusion in sentence form.

Exercise

A

1. If x is a number, state

(a) twice the number, (b) the number increased by 3,
(c) 5 less than the number, (d) 11 decreased by the number,
(e) one half the number increased by itself,
(f) the number which added to x makes 13.

2. If n is an integer, state

(a) the next consecutive integer, (b) the previous integer,
(c) the sum of the three integers of which n is the middle one.

3. If the width of a rectangle is w inches and the length is 4 inches more than the width, state

(a) the length, (b) the perimeter.

4. If John is x years old and Mary is twice as old, state

(a) John's age 5 years ago, (b) Mary's present age,
(c) John's age 7 years from now, (d) Mary's age 5 years ago,
(e) The sum of their present ages,
(f) The sum of their ages three years from now.

5. If I have x nickels and three times as many dimes in my pocket, state

(a) the number of dimes, (b) the value of the dimes,
(c) the value of the nickels, (d) the total value of the coins,
(e) the total number of coins.

B

6. When 5 is added to twice a certain number, the sum is 17. What is the number?

7. When a number is multiplied by 3, and 5 is added to the product, the result is 26. What is the number?

8. Three times a number, decreased by 2, is equal to twice the same number increased by 7. Find the number.

9. If you add 12 to a certain number, the result is the same as if you added 4 to twice the number. Find the number.

10. Find two consecutive integral numbers the sum of which is 51.

11. Find three consecutive integral numbers the sum of which is 63.

12. Four times a number is equivalent to the number increased by 51. Find the number.

13. One quarter of a number, increased by one half the same number, is 12. Find the number.

14. Five per cent of a certain number is 22. Find the number.

15. A number increased by 30% of itself is 420. Find the number.

16. During the summer, darkness is 5 hours less than daylight. How many hours of darkness are there on this particular day?

17. If the two equal sides of an isosceles triangle are each 4 times as long as the base, and the perimeter is 108 inches, find the lengths of the three sides.

18. The perimeter of an equilateral triangle is 72 inches. Find the length of each side.

19. A board 12 ft. long is to be cut into 3 pieces so that each section is 4 in. longer than the previous one. What is the length of the shortest piece?

20. A mother is three years less than 5 times as old as her daughter. If the sum of their ages is 39, find the age of each.

21. A man has the same number of dimes as he has nickels in his pocket. If he has $2.85, how many of each has he?

22. A father is 19 years older than his son. If the sum of their ages is 31, find the age of each.

23. A man has in his pocket four times as many dimes as he has nickels. If the value of his change is $2.70, how many of each coin has he?

24. A bottle filled with varnish costs $1.10. If the varnish is worth $1 more than the bottle, find the value of the bottle. Be sure to check your solution.

25. In a certain triangle, one side is 6 inches longer than the base and the other side is 5 inches longer than the base. If the perimeter is 131 inches, how long is each side?

26. A house and lot cost $10,000. The house costs $500 more than three times the cost of the lot. Find the cost of each.

27. In selling a tie for $2, a merchant makes a profit of 10% on the cost price. Find the cost price of the tie.

28. Divide an estate of $4,500 so that the wife's share will be $400 more than three times that of her only son.

29. In selling a piece of costume jewellery for $4.60, the jeweller makes a profit of 15% on the cost price. Find the cost price.

30. In selling a saw for $77, the hardware merchant makes a profit of 40% on the cost price. Find the cost price.

31. A man left an estate of $167,500 to be disposed of as follows: $2,500 to charity, $3,000 to education and the remainder to be divided so that his widow would get ten times as much as each son, which is fives times as much as each daughter. If there were two sons and three daughters, how much did each get?

32. A man developed a rare variety of wheat. For each bushel planted, he harvested 10 bushels. Then he planted his entire crop and again got a 10-fold yield. He repeated this once more with a 10-fold gain and this time harvested 500 bushels. How many bushels did he plant the first time?

33. A building 4,752 feet away from an observer was struck by lightning. It took approximately $4\frac{2}{5}$ seconds for the sound to reach the observer's ear. Calculate the rate at which sound travels.

C

34. Solve:
(*a*) $3x - 7 = 2x + 8$, (*b*) $2x - 5 = 8 - x$,
(*c*) $5x - 1 = 3x + 13$, (*d*) $0.3x - 0.6 = 0.2x$,
(*e*) $0.5x - 1 + 0.2x = 5 - 0.3x$, (*f*) $\frac{1}{4}x + \frac{1}{2} = \frac{1}{3}x - \frac{1}{4}$.

35. The sum of three consecutive numbers is 54. Find the numbers.

36. The length of a rectangle is 2 feet more than the width, and the perimeter is 76 ft. Find the length and the width.

37. The sides of a triangle are x feet, $(x+6)$ feet and $(2x+5)$ feet respectively. If the perimeter is 91 feet, find the length of each of the sides.

38. A man is 25 years older than his son and 27 years older than his daughter. If the man and his wife are the same age, and the total age of the family is 76 years, find the age of each person.

Review Exercise

A

1. State in words the axioms used in solving each of the following:
(*a*) $x + 8 = 24$. (*b*) $y - 6 = 7$. (*c*) $3y = 8$. (*d*) $\frac{1}{4}y = 6$.

2. Solve and verify:
(*a*) $3x + 6 = 24$. (*b*) $3x - 5 = 16$.
(*c*) $5x - 8 = 3x$. (*d*) $6 - x = x$.

B

3. Solve and verify:
(*a*) $6x + 7 = 2x + 15$. (*b*) $8x - 3 = 5x + 27$.
(*c*) $2x - 1 = 4x - 7$. (*d*) $x - 3 = 2x - 1$.

4. Solve and verify:
(*a*) $6x - 2 = x - 1$. (*b*) $15x + 3 = 4x + 4$.
(*c*) $3x - \frac{1}{4} = \frac{1}{4} - x$. (*d*) $14a - 3a = a + 0{\cdot}01$.

5. When double a number is increased by 6, the result is 17. Find the number.

6. The sum of four consecutive numbers is 130. Find the numbers.

7. A wire 64″ long is bent into a rectangle the length of which is 2″ more than the width. Find the dimensions.

8. Divide \$27,000 into three parts so that the second part is three times the first and the third part is fives times the first.

9. If $h = \frac{2A}{b}$ and $h = 16$ when $b = 12$, find A.

10. One line is 3″ shorter than twice the length of another line. If their total length is 81″, find the length of each.

11. A contractor estimates that he will use 1 part of cement, 2 parts of sand and 4 parts of gravel for each batch of concrete. If he wishes to make enough concrete to fill 100 cu. ft., how many bags of cement will he order? 1 bag of cement measures 1 cubic foot. Assume the shrinkage in the materials is equivalent to the amount of cement used.

C

12. Solve each of the following equations and verify your results:

(a) $\frac{2}{3}s - 8 = 4$, (b) $\frac{5}{4}t + 10 = 0$,

(c) $\frac{3}{4}a - 2 = \frac{1}{2}a + 3$, (d) $12 + \frac{3}{4}x = 0$,

(e) $\frac{3n + 1}{5} = 2$, (f) $1.3t + 5.14 = 2.1 - 1.9t - 9.76$.

13. Solve for x or y and verify your results:

(a) $x + m = n$, (b) $2y - p = 6$,

(c) $9r^2 = 6r - 3rx$, (d) $3ay - ab + 2ay = ac$,

(e) $5ay - 10 = 2ay$, (f) $\frac{x - m}{4} = m$.

14. The sum of $\frac{1}{2}$ the number, $\frac{1}{4}$ of the number and the number is 14. Find the number.

15. A boy receives a sum of money for his 16th birthday. He spends $\frac{1}{3}$ of it for a set of tools, $\frac{1}{4}$ of it for a book, and deposits the remainder, amounting to \$20.00 in his savings account. How much did he receive as a birthday gift?

16. If 8 pencils cost 24 cents more than 5 of the same quality of pencils, what is the cost of each pencil?

17. A man was hired for 20 days. Each day he worked he received \$10.00 plus board. For every day he did not work he was charged \$3.00 for board. If he received \$161.00 in all, how many days did he work?

CHAPTER 13

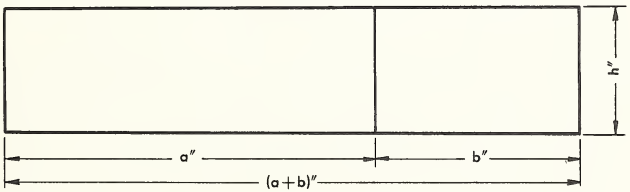
Operations with Algebraic Expressions

Measures expressed algebraically may have one, two or more terms. One-term expressions such as 25, 3×6, $9 \div 3$, $3ax$, $-x$, $-ay^3$, etc. are called **monomials**. Two-term expressions such as $x + y$, $5 \times 3 - 2 \times 4$, $3a - 3b$, $2x - 5$, etc. are called **binomials**. Three-term expressions such as $a + b + c$, $x - 2y + 5$ and $x^2 - 3x + 4$ are called **trinomials**.

Multiplication by a Monomial

The following problem will illustrate the method by which an algebraic expression is multiplied by a monomial factor.

Find the area of a rectangle the length of which in inches is $a + b$ and the width of which in inches is h.



The area of the first small rectangle in square inches is ha.

Area of second small rectangle in square inches is hb.

Area of the large rectangle in square inches is $ha + hb$.

But the area of the large rectangle in square inches is $h(a + b)$

$$\therefore h(a + b) = ha + hb.$$

To multiply an expression of more than one term by a monominal, multiply each term of the expression by the monomial factor.

Example 1: Perform the indicated multiplications:

(*a*) $3a^2(a + c)$, (*b*) $4x(x - y + 1)$.

Solutions:

(*a*) $3a^2(a + c)$
$= 3a^3 + 3a^2c.$

(*b*) $4x(x - y + 1)$
$= 4x^2 - 4xy + 4x.$

Example 2: Express in simplified form the expression

$$3(a - b) + 4(b + c) - 5(c - a).$$

The simplification of this involves the operations of multiplication, addition and subtraction. According to *the order of operations*, multiplication is performed before the addition or the subtraction.

Solution:

$3(a - b) + 4(b + c) - 5(c - a)$	Perform the indicated multiplication.
$= (3a - 3b) + (4b + 4c) - (5c - 5a)$	Remove the brackets.
$= 3a - 3b + 4b + 4c - 5c + 5a$	Collect like terms.
$= 8a + b - c.$	

Exercise

A

1. Expand each of the following products:

(*a*) $5(a - b)$, (*b*) $x(x + 6)$,
(*c*) $3a(2a^2 + 6a + 3)$, (*d*) $-2(a - 3b + 5c)$,
(*e*) $-3s(4r + 2s)$, (*f*) $-2xy(3x^2 - 3xy + 2y)$.

2. Expand the following products:

(*a*) $x(x - 1)$, (*b*) $\pi(R^2 - r^2)$,
(*c*) $\frac{1}{2}(6x - 4y)$, (*d*) $-7(3 - 2a)$,
(*e*) $2(-4b - 3c)$, (*f*) $-3ab(2a - 3b + a^2)$.

3. Express the area of each of the following rectangles and find each product:

(*a*) length in feet $x + y$ and width in feet 5;
(*b*) length in inches $a - 3b$ and width in inches $4a$;
(*c*) length in inches $3x + 2y - z$ and width in feet $\frac{1}{2}x$.

4. What is the volume of a square prism if the length of each side of the base in inches is $2a$ and the height in inches is $3a - 5b$?

B

5. Find each of the following products:

(*a*) $3(a - b + c)$, (*b*) $5(x + y + 1)$,
(*c*) $2x(3a - 4b + 5)$, (*d*) $2x(x^2 + 3x + 1)$,
(*e*) $4y(y^2 + 3y - 1)$, (*f*) $\frac{1}{2}x(6x^2 - 10x - 12)$,
(*g*) $3x^2y(xy - 3)$, (*h*) $\frac{1}{2}h(x + y)$.

6. Simplify each of the following expressions:

(*a*) $2(x + y) - 3(x - y) + 5(x + y)$;
(*b*) $3x(x^2 + 5x - 1) - x(2x + 3) - 4(x - 5)$;
(*c*) $4(2a - b) - 3(3a - 5b) + 4(a - 2b)$;
(*d*) $xy(xy - 1) + xy(xy + 1)$;
(*e*) $3ab(a - b) + 2ab(b - a)$.

7. Express the difference in the circumferences of the two circles in the diagrams below. Simplify the resulting expression.

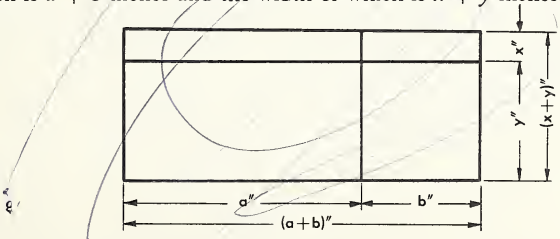
8. Simplify:
   (*a*) $5(x + 2) - 3(x - 1)$; (*b*) $3(6 - x) - 4(x + 8)$;
   (*c*) $2a - (1 - 3a) + 3(1 + a)$; (*d*) $z(z - 1) - z(3 - z)$.
9. Simplify:
   (*a*) $3(x^2 - 3x + 2) - 5(2x^2 + 3x - 3)$;
   (*b*) $2x(x^2 - 3x + 1) + 3(x^2 + 2x - 4)$;
   (*c*) $-2(a - 3b + c) + 3(2a - 3b + c)$;
   (*d*) $-2y^2(y - 3) - 3y(y - y^2) - 2(y^2 - 2)$.
10. Subtract the product of x and $x + y$ from the product of y and $x + y$.
11. Subtract the product of -3 and $2x - 3y + 4$ from 20.

C

12. If $x = a + b$ and $y = a - c$ and $z = b - c$, find the value of
    (*a*) $2x + 3y - 5z$, (*b*) $cx + by + az$.
13. If $y = 3x - 2$ and $z = 5x - 1$, find the value of $6y - 2z$.
14. If $A = \pi d$, $B = 3\pi d$, $C = \pi d - 3$, find the value of $6A - B - 2C$.

Multiplication by a Binomial

Examine the sketch to determine the area of the rectangle the length of which is $a + b$ inches and the width of which is $x + y$ inches.



From the diagram, we see that the area of the rectangle is the sum of the areas of the four smaller rectangles. Hence, the area in square inches is $ax + bx + ay + by$.

But the area of the rectangle in square inches is $(x + y)(a + b)$.

$$\therefore (x + y)(a + b) = ax + bx + ay + by.$$

It will be observed that the expression on the right is obtained by multiplying each term of the second factor, $a + b$, by each term of the first factor, $x + y$, and combining the resulting *partial products* by addition.

Similarly, to *expand* $(x - a)(x - b)$, we multiply each term of the second factor by each term of the first factor using the proper sign with each *partial product* and obtain

$$(x - a)(x - b) = x^2 - xb - ax + ab.$$

To expand the product of two binomials, multiply each term of one factor by each term of the other factor and add or subtract the partial products as indicated by their signs.

Example 1: Expand the product $(a - 5)(a + 7)$.

Method 1: Horizontal arrangement

$$(a - 5)(a + 7)$$
$$= a^2 + 7a - 5a - 35$$
$$= a^2 + 2a - 35.$$

Method 2: Column arrangement

$$\begin{array}{r} a - 5 \\ a + 7 \\ \hline a^2 - 5a \\ \quad + 7a - 35 \\ \hline a^2 + 2a - 35 \end{array}$$

$$\therefore (a - 5)(a + 7) = a^2 + 2a - 35.$$

Example 2: Perform the indicated multiplication $(2x + 3)(3x - 2)$.

Method 1:

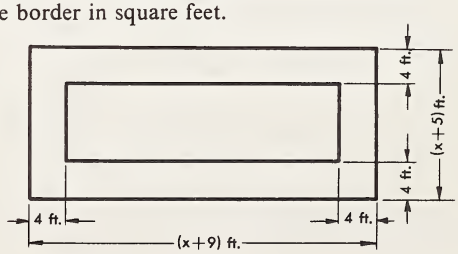
$$(2x + 3)(3x - 2)$$
$$= 6x^2 - 4x + 9x - 6$$
$$= 6x^2 + 5x - 6.$$

Method 2:

$$\begin{array}{r} 2x + 3 \\ 3x - 2 \\ \hline 6x^2 + 9x \\ \quad - 4x - 6 \\ \hline 6x^2 + 5x - 6 \end{array}$$

$$\therefore (2x + 3)(3x - 2) = 6x^2 + 5x - 6.$$

Example 3: The length and the width of a lot expressed in feet is $x + 9$ and $x + 5$ respectively. A border 4 feet wide is placed within this lot. Express the area of the border in square feet.



The area of the outside rectangle in sq. ft. is $(x + 5)(x + 9)$.
The length of the inside rectangle in feet is $x + 9 - 2 \times 4$
$= x + 1.$
The width of the inside rectangle in feet is $x + 5 - 2 \times 4$
$= x - 3.$
The area of the inside rectangle in sq. ft. is $(x + 1)(x - 3)$.
The area of the border in sq. ft. is $(x + 5)(x + 9) - (x + 1)(x - 3)$
$= (x^2 + 9x + 5x + 45) - (x^2 - 3x + x - 3)$
$= (x^2 + 14x + 45) - (x^2 - 2x - 3)$
$= x^2 + 14x + 45 - x^2 + 2x + 3$
$= 16x + 48.$

Exercise B

1. Use the horizontal arrangement to expand each of the following products:

(*a*) $(x + 5)(x + 3)$, (*b*) $(y + 9)(y + 2)$,
(*c*) $(x - 3)(x - 2)$, (*d*) $(a - 7)(a - 5)$,
(*e*) $(3x + 3)(3x + 5)$, (*f*) $(4a - 3)(5a - 2)$,
(*g*) $(x - 2)(x + 3)$, (*h*) $(x + 2)(x - 3)$.

2. Use the column arrangement to expand each of the following products:

(*a*) $\begin{array}{r} x + 4 \\ x + 5 \\ \hline \end{array}$ (*b*) $\begin{array}{r} x - 3 \\ x - 6 \\ \hline \end{array}$ (*c*) $\begin{array}{r} x + 7 \\ x - 9 \\ \hline \end{array}$

(*d*) $\begin{array}{r} 2x - 4 \\ 4x - 7 \\ \hline \end{array}$ (*e*) $\begin{array}{r} 5a + 4b \\ a - 9b \\ \hline \end{array}$ (*f*) $\begin{array}{r} 6a - 5b \\ 4a + 7b \\ \hline \end{array}$

3. Expand the following expressions:

(*a*) $(x + y)(a + b)$, (*b*) $(a + b)(x - y)$, (*c*) $(a - b)(x - y)$,
(*d*) $(1 - n)(5 - n)$, (*e*) $(q + 5)(q - 5)$, (*f*) $(a + 4b)(a - 3b)$.

4. Expand the following products:

(*a*) $(x + y)(x + y)$, (*b*) $(5a - 3b)(5a - 3b)$, (*c*) $(x + \frac{1}{2})(x + \frac{1}{2})$.

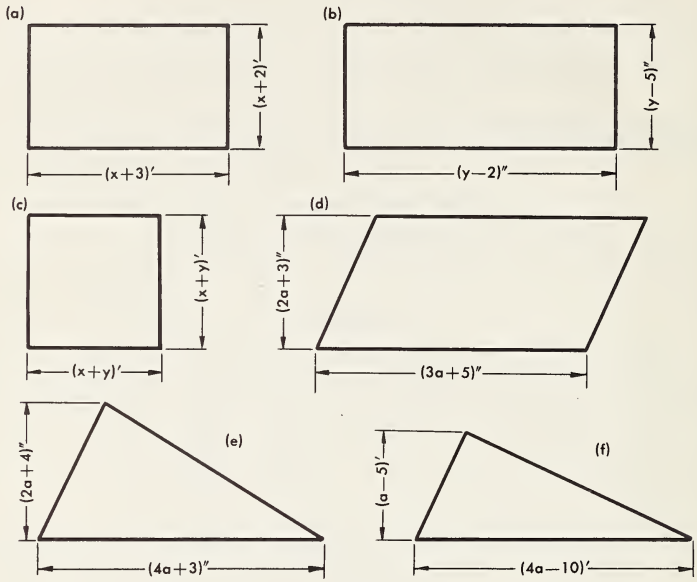
5. Expand the following:

(*a*) $(x + 4)^2$, (*b*) $(x - 3)^2$,
(*c*) $(m + n)^2$, (*d*) $(m - n)^2$,
(*e*) $(2 + y)^2$, (*f*) $(5 - y)^2$,
(*g*) $(2x + 5)^2$, (*h*) $(3x - 7)^2$.

6. Perform the indicated multiplication:

(*a*) $(x + 3)(x - 3)$, (*b*) $(a + 5)(a - 5)$,
(*c*) $(x - y)(x + y)$, (*d*) $(a + b)(a - b)$,
(*e*) $(3 - x)(3 + x)$, (*f*) $(3 - 5y)(3 + 5y)$.

7. Find the area of each of the following figures:



8. Simplify:

(*a*) $(2x + 5)(3x + 4) + (x + 1)(x + 3)$,
(*b*) $(3x - 4)(2x - 1) - (2x - 3)(4x + 1)$,
(*c*) $(x + y)^2 - (x - y)^2$,
(*d*) $(3x - 4y)(3x + 4y) + (2x - y)(2x + y)$.

9. Express and simplify the difference between the product $(x + 5b)(x - 3b)$ and the product $(x - 5b)(x + 5b)$.

10. Find the difference in area between two rectangles, the first of which has dimensions of $3x + 5y$ and $4x - 3y$ in linear units and the second of which has dimensions of $4x + 3y$ and $2x - y$ in the same linear units.

11. Simplify:

(*a*) $(x + 1)(x - 2) + 5 - (1 - x)(x - 2)$;
(*b*) $3(n - 1)(n - 2) - (3n - 1)(n - 1)$;
(*c*) $(y + 1)^2 + y^2 - 1 - (y - 1)^2$;
(*d*) $(x - 2)(2 - x) + 2(x - 2)(x + 2) - 3(x - 1)^2$.

12. Simplify:

(*a*) $3(x - y) - 5(2x - 3y) + 4(x - 3y)$;
(*b*) $2x(x - 4y) - 5y(2x - y) + (2x - y)^2$;
(*c*) $(4x - 3y)^2 - (4x + 3y)^2 + 2(4x - 3y)(4x + 3y)$.

C

13. Express in cubic feet the volume of a rectangular prism 8 feet in height if the length and the width of the base in feet is $x + 3$ and $x - 5$ respectively.

14. What is the volume of a square prism 3 feet high if each of the sides of the base measures $x + y$ feet?

15. Express the area of a circle if the length of the radius in feet is $a + b$. Expand the expression.

16. Express the area of a triangle if the length of the base in feet is $x + 2$ and the height in feet is $x - 2$.

Sight Multiplication

With a little thought and practice, the product of two binomials may be written down at sight.

In the product $(x + 5)(x - 3)$, the first term of the product is x^2. The middle term is made up of the two *cross-products* $5x$ and $-3x$, hence the middle term is $+2x$. The last term is -15.

$$\therefore (x + 5)(x - 3) = x^2 + 2x - 15.$$

$$\begin{array}{r} x + 5 \\ x - 3 \\ \hline x^2 + 5x \\ -3x - 15 \\ \hline x^2 + 2x - 15 \end{array}$$

Similarly $(2x + 5)(3x - 2) = 6x^2 + 11x - 10$.

Special Types of Products of Two Binomials

(1) **The Square of a Binomial**

$$\begin{aligned} (x + y)^2 &= (x + y)(x + y) \\ &= x^2 + 2xy + y^2. \\ (x - y)^2 &= x^2 - 2xy + y^2. \end{aligned}$$

The square of a binomial is equal to the sum of the square of the first term, twice the product of the first and second terms with the proper sign and the square of the last term.

Example: Expand $(3x + 4y)^2$ and $(5a - 3b)^2$.

Solution:
$$\begin{aligned} (3x + 4y)^2 &= (3x)^2 + 2(3x)(4y) + (4y)^2 \\ &= 9x^2 + 24xy + 16y^2. \\ (5a - 3b)^2 &= (5a)^2 - 2(5a)(3b) + (3b)^2 \\ &= 25a^2 - 30ab + 9b^2. \end{aligned}$$

(2) **Product of the Sum and Difference of the Same Two Numbers**

$$(2a + 5)(2a - 5) = 4a^2 - 25.$$
$$(2 - 3x)(2 + 3x) = 4 - 9x^2.$$
$$(a + b)(a - b) = a^2 - b^2.$$

The product of the sum and difference of two numbers is the same as the difference of their squares.

Exercise

A

Expand the following expressions:

1. $(x + 2)(x + 3)$,
2. $(x + 7)(x + 5)$,
3. $(a + 4)(a + 3)$,
4. $(a + 9)(a + 1)$,
5. $(x + 5)(x + 6)$,
6. $(x + 8)(x + 1)$,
7. $(2x + 3)(x + 5)$,
8. $(2x + 5)(x + 4)$,
9. $(2x + 3)(2x + 5)$,
10. $(2x + 5)(3x + 7)$,
11. $(x - 3)(x - 2)$,
12. $(x - 5)(x - 4)$,
13. $(x - 1)(x - 3)$,
14. $(x - 2)(x - 9)$,
15. $(a - 3)(a - 7)$,
16. $(a - 1)(a - 10)$,
17. $(2a - 3)(3a - 1)$,
18. $(3a - 2)(2a - 7)$,
19. $(4a - 3)(2a - 5)$,
20. $(5a - 1)(a - 2)$,
21. $(x + 3)(x - 4)$,
22. $(x - 3)(x + 4)$,
23. $(x - 7)(x + 3)$,
24. $(x - 3)(x + 7)$,
25. $(a - 11)(a + 1)$,
26. $(a + 7)(a - 9)$,
27. $(3a - 2)(2a - 3)$,
28. $(2a - 3)(2a + 5)$,
29. $(5x - 3y)(5x + 7y)$,
30. $(5a - b)(a - 5b)$,
31. $(a + b)^2$,
32. $(a + 4)^2$,
33. $(a + 5)^2$,
34. $(a + 7)^2$,
35. $(x - 3)^2$,
36. $(x - 6)^2$,
37. $(x + 2y)^2$,
38. $(2x - y)^2$,
39. $(2a - 3)^2$,
40. $(b - c)^2$,
41. $(3a + b)^2$,
42. $(3 - 2y)^2$,
43. $(2 - x)^2$,
44. $(2xy - 5)^2$,
45. $(x + \frac{1}{2})^2$,
46. $(x - 1)^2$,
47. $(a + 4)(a - 4)$,
48. $(x - 3)(x + 3)$,
49. $(a + 7)(a - 7)$,
50. $(x - y)(x + y)$,
51. $(a - 11)(a + 11)$,
52. $(2x - 3)(2x + 3)$,

53. $(3x - 5)(3x + 5)$, **54.** $(3x - 2y)(3x + 2y)$,

55. $(x - 1)(x + 1)$, **56.** $(4x + 5y)(4x - 5y)$.

B

Write the squares of the following:

57. $a + 8$, **58.** $b - 7$, **59.** $3 - a$,

60. $4c - d$, **61.** $-2p - 2q$, **62.** $-8 - x$,

63. $4y + \frac{1}{2}$, **64.** $a + \frac{1}{a}$, **65.** $b - \frac{1}{b}$,

66. $3ab - 2c$, **67.** $1 + x^2$, **68.** $2x + x^2$.

Expand the following:

69. $(1 + x^2)(1 - x^2)$, **70.** $(2a - b)(b - 2a)$,

71. $(3 + x)(x + 3)$, **72.** $(a + b)(b + c)$,

73. $(a - b)(x + y)$, **74.** $(x + 2y)(2x - y)$.

Simplify:

75. $3(x + 2)^2 + 2(x - 3)^2$; **76.** $3(x - 2)(x + 4) - (x + 4)(x - 4)$;

77. $(x + y)(x - y) - (x - y)^2$; **78.** $(a + b)^2 + (a - b)^2$;

79. $(4x - 3y)(4x + y) - (3x - 2y)(2x - 5y)$;

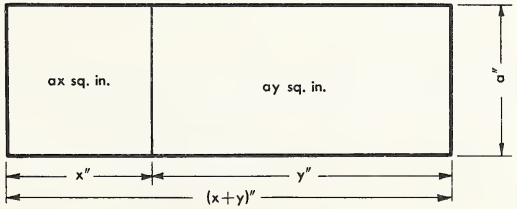
80. $2(a + b)^2 + 3a(a - 2b) - 4(a - 2b)^2$;

81. $2 - 3x + 5x^2 - (x + 5)(2x - 3)$.

Division by a Monomial

What is the length of a rectangle which has an area of $(ax + ay)$ square inches and a width of a inches.

Since the area is $ax + ay$ square inches, this rectangle can be considered to be the sum of two smaller rectangles, the areas of which are ax square inches and ay square inches respectively.



The length in inches is expressed as $\frac{ax + ay}{a}$.

But from the diagram the length in inches is $x + y$.

$$\therefore \frac{ax + ay}{a} = x + y.$$

The quotient when $ax + ay$ is divided by a is obtained by dividing each term of the dividend by the divisor. This division may be checked by multiplying the quotient $x + y$ by the divisor a.

Similarly, to divide $3x^3 + 6x^2 + 12x$ by $3x$ divide each term of the dividend by the divisor.

$$\therefore \frac{3x^3 + 6x^2 + 12x}{3x} = x^2 + 2x + 4.$$

To divide an expression of more than one term by a monomial, each term of the dividend must be divided by the divisor.

Example: Divide $2a^3 - 8a^2 - 6a$ by $-2a$.

Solution: $\dfrac{2a^3 - 8a^2 - 6a}{-2a} = -a^2 + 4a + 3.$

Exercise

A

Divide:

1. (*a*) $+6ax$ by $-2a$, (*b*) x^5 by x^2, (*c*) $-6xy$ by $-3y$, (*d*) $-a$ by a.

2. $\dfrac{bx - by}{b}$,

3. $\dfrac{4a + 4}{4}$,

4. $\dfrac{6x^2 - 3x^3 + 12x}{3x}$,

5. $\dfrac{ax + ay - az}{a}$,

6. $\dfrac{12a^2 - 8a^2c + 20ac^2}{4a}$,

7. $\dfrac{x + y}{-1}$,

8. $\dfrac{4x^3 + 6x^2 + 8x}{-2x}$,

9. $\dfrac{10a^2b - 5ab}{-5ab}$.

B

Perform the indicated divisions:

10. $\dfrac{6a + 3b}{3}$,

11. $\dfrac{6x^2 - 15x - 18}{-3}$,

12. $\dfrac{5a^2b - 10ab^2}{-5ab}$,

13. $\dfrac{6x^3 - 12x^2 - 9x}{3x}$,

14. $\dfrac{3a + 2b}{0{\cdot}1}$,

15. $\dfrac{1{\cdot}5a - 1{\cdot}8}{0{\cdot}3}$,

16. $\dfrac{a^3 - a^2 - a}{-a}$,

17. $\dfrac{cx + c}{c}$.

Simplify:

18. $\frac{2x+4}{2}+\frac{3x-9}{3}$;

19. $\frac{ax+ay}{a}-\frac{bx-by}{b}$;

20. $\frac{x^2+x}{x}-\frac{x^2-x}{x}$;

21. $\frac{16ab-20ac}{-4a}+\frac{12bc-15ac}{3c}$;

22. $3(a+2b)-\frac{3a-6b}{3}$;

23. $\frac{5a^2-15ab}{5a}-\frac{3a^2-12a}{3a}$.

C

24. Two different rectangles each have an area of $6x^2 - 12x$ square inches. If the width of one is $2x$ inches and the width of the other is $3x$ inches, express the difference in their lengths and simplify the expression.

25. Each of the measures below expresses in square inches the area of a rectangle. Find suitable measures for the length and the width of each rectangle. Arrange your results in a table similar to the one below.

(a) $3x + 3y$.
(b) $ab + ac$.
(c) $5x - 10$.
(d) $x^2 - x$.
(e) $3x^2 - 6x$.
(f) $bc - ab$.
(g) $xy + y$.
(h) $8 - 4y$.
(i) $x^2 + 5x + 6$.
(j) $x^2 + 3x + 2$.

Area in square inches	*Length in inches*	*Width in inches*
$2x + 2y$	2	$x + y$

Division by a Binomial

The area of a rectangular lot is $(x^2 - x - 6)$ sq. ft. The length of the lot is $(x + 2)$ ft. Find the width of the lot.

Solution: The number of feet in the width is $\frac{x^2-x-6}{x+2}$

To evaluate this measure it is required that the operation of division by a binomial be performed.

$$
\begin{array}{r}
x-3 \\
x+2\ \overline{)\,x^2-x-6} \\
\underline{x^2+2x} \\
-3x-6 \\
\underline{-3x-6} \\
0
\end{array}
$$

The width is $(x - 3)$ feet.

The procedure is as follows:

1. The divisor and the dividend are arranged in descending or ascending order of powers of the variable.
2. Divide the first term of the divisor into the first term of the dividend *exactly*, getting the first term of the quotient.
3. Multiply the divisor by the term of the quotient just obtained and subtract the product from the dividend.
4. Divide the first term of the divisor into the first term of the new dividend *exactly*, to get the second term of the quotient.
5. Repeat steps 3 and 4 until a remainder of zero or of a degree lower than that of the divisor is obtained.

Example: Divide $x^3 - 27$ by $x - 3$

Hint: The dividend may be written as $x^3 + 0x^2 + 0x - 27$

Solution:

$$
\begin{array}{rl}
 & x^2 + 3x + 9 \\
x - 3 \;) & \overline{x^3 + 0x^2 + 0x - 27} \\
 & \underline{x^3 - 3x^2} \\
 & \quad 3x^2 + 0x \\
 & \quad \underline{3x^2 - 9x} \\
 & \qquad 9x - 27 \\
 & \qquad \underline{9x - 27} \\
 & \qquad\quad 0
\end{array}
$$

Thus $\dfrac{x^3 - 27}{x - 3} = x^2 + 3x + 9.$

Exercise

1. Divide:
   (*a*) $2x^2 - 3x - 2$ by $x - 2$, (*b*) $x^2 + 2x - 35$ by $x + 5$,
   (*c*) $x^2 - 9$ by $x - 3$, (*d*) $6a^2 + 7a + 2$ by $2a + 1$,
   (*e*) $12n^2 - 11n - 5$ by $4n - 5$, (*f*) $2x^2 - 13x + 20$ by $2x - 5$.

Perform the indicated divisions:

2. $\dfrac{x^3 - 8}{x - 2}$,
3. $\dfrac{x^2 + 2xy + y^2}{x + y}$,
4. $\dfrac{5x^2 - 13x - 6}{x - 3}$,
5. $\dfrac{a^2 + 6a + 9}{3 + a}$,
6. $\dfrac{2x^2 + 3xy - 2y^2}{2x - y}$,
7. $\dfrac{4 - 10x + 4x^2}{4 - 2x}$.

Divide:

8. $m^2 - 5m + 6$ by $-2 + m$,
9. $x^3 + 3x^2 - 2x - 8$ by $x + 2$,
10. $x^3 - 10x + 2x^2 - 8$ by $4 + x$,
11. $4k^2 + 2k + 1 + 2k$ by $2k + 1$.

Review Exercise

A

1. Expand:

(*a*) $(x + 4)(x - 4)$, (*b*) $(2x + 3)(2x - 3)$,
(*c*) $(2x - 3y)(2x + 3y)$, (*d*) $(2a + 5b)(2a - 5b)$,
(*e*) $(3a - 7b)(3a + 7b)$, (*f*) $(5a - 6xy)(5a + 6xy)$.

2. The length of a rectangle is $3x - 5y$ inches and the width is $5x$ inches. Find its area.

3. Simplify:

(*a*) $(x + 2)(x + 6)$, (*b*) $(x + 4)(x - 3)$,
(*c*) $(x - 8)(x + 8)$, (*d*) $(2x + 3)(3x - 4)$,
(*e*) $(5x - 4)^2$, (*f*) $(5x + 6)^2$.

4. The area of a rectangle is $12a^2 - 3a$ square inches. If the width is $3a$ inches, find the length.

5. Perform the divisions indicated:

(*a*) $\dfrac{6a + 4b}{2}$, (*b*) $\dfrac{5x^2 - 15}{-5}$,

(*c*) $\dfrac{x^2 + 6x}{x}$, (*d*) $\dfrac{2x^2 - 8x}{-2x}$,

(*e*) $\dfrac{3a^2b - 5ab^2}{ab}$, (*f*) $\dfrac{3x^3 - 6x^2}{-3x^2}$.

B

6. Simplify:

(*a*) $5(a - 3)$; (*b*) $2(a - 1) + 4(a - 2)$;
(*c*) $4(x - y) - 3(x + y)$; (*d*) $(a - b) - (a + 2b)$.

7. Find the volume of a rectangular prism which has a length of $x + 4$ inches, a width of $x + 3$ inches and a height of 4 inches.

8. Perform the multiplication that is indicated:

(*a*) $3(x - 1)(x + 3)$, (*b*) $5x(x - 1)^2$,
(*c*) $2(a - 3b)(2a + b)$, (*d*) $-3(2x - 5y)(2x + 5y)$.

9. Simplify:

(*a*) $(x + 3)(x + 4) + (x + 2)(x + 1)$;
(*b*) $(x + 6)(x - 2) - (x + 7)(x - 3)$;
(*c*) $(x + 3)^2 - (x - 3)^2$;
(*d*) $(a + 5b)(a - 5b) + (2a - 3b)(a + 7b)$;
(*e*) $3(x + 1)^2 + 2(x + 3)^2$;
(*f*) $2(x - 7)^2 - 3(x - 6)^2$.

10. Simplify:

(*a*) $\frac{15}{3} - \frac{8}{4}$;

(*b*) $\frac{5x}{x} - \frac{3y}{y}$;

(*c*) $\frac{3x + 24y}{3} + \frac{5x + 20y}{5}$;

(*d*) $\frac{12x - 3y}{3} - \frac{8x + 12y}{4}$.

C

11. Simplify:

(*a*) $(2x - 3)^2 - (x - 2)^2$,

(*b*) $(x + 4)(x + 3) - (x - 1)^2$,

(*c*) $4x(x + 5) + 4(x + 1)(x - 2) - 2(x - 2)^2$.

12. Divide:

(*a*) $a^3 + b^3$ by $b + a$,

(*b*) $x^3 + 3x^2 + 3x + 1$ by $x + 1$,

(*c*) $x^3 - 3x^2 + 3x - 1$ by $x - 1$.

13. Divide:

(*a*) $2y^2 + y - 6$ by $2y - 3$,

(*b*) $3x^2 - 4x - 4$ by $x - 2$,

(*c*) $y^2 + y - 2$ by $y + 2$,

(*d*) $y^2 - y - 6$ by $y - 3$,

(*e*) $a^2 + 3a - ab - 3b$ by $a - b$,

(*f*) $x^2 - 4y^2$ by $x - 2y$.

14. The area of a rhombus is $(2m^2 + m - 3)$ sq. units. If the altitude is $(m - 1)$ units, find the length of the side.

15. The area of a triangle is $(3x^2 - 10x + 3)$ sq. ins. If the base is $(3x - 1)$ ins. long, find the length of the altitude.

CHAPTER 14

Factoring

Since $b(x + y) = bx + by$ then, of course, the factors of $bx + by$ are $b(x + y)$. In the first instance, the operation of multiplication was carried out with the two factors b and $(x + y)$ to obtain the expression $bx + by$. In the second the operation of factoring was carried out to obtain the factors of the expression $bx + by$.

Our first method of factoring expressions is to think, "*What factors were multiplied together to give this expression?*" Then, having decided the factors, multiply them and check with the original expression.

Simple Factors

In arithmetic, if 30 is factored we may obtain different sets of factors: $30 = 3 \times 10$; $30 = 5 \times 6$; or $30 = 2 \times 3 \times 5$. None of the numbers in the last set of factors can themselves be factored. Thus 2, 3 and 5 are the prime factors of 30.

In algebra $5xy$ may be expressed as a product of different sets of factors also:

$$5xy = 5x \times y; \qquad 5xy = 5 \times xy;$$
$$5xy = 5y \times x; \qquad 5xy = 5 \times x \times y.$$

None of the factors in the last set can themselves be factored. Thus 5, x and y are the **simple factors** of $5xy$.

Example 1: Express as a product of its simple factors (*a*) $12xy$, (*b*) $15x^2y$.

Solution:

(*a*) $12xy = 2 \times 2 \times 3 \times x \times y$. (*b*) $15x^2y = 3 \times 5 \times x \times x \times y$.

Example 2: What are the simple factors of (*a*) $3x - 3y - 3z$, (*b*) $x^2 + 5x + 6$, (*c*) $x^2 + 6xy + 9y^2$, (*d*) $x^2 - y^2$?

Solution:

(*a*) $3x - 3y + 3z = 3(x - y + z)$ since the product of 3 and $x - y + z$ is $3x - 3y + 3z$.

(*b*) $x^2 + 5x + 6 = (x + 2)(x + 3)$ since the product of $x + 3$ and $x + 2$ is $x^2 + 5x + 6$.

(*c*) $x^2 + 6xy + 9y^2 = (x + 3y)(x + 3y)$ or $(x + 3y)^2$ since $(x + 3y)^2 = x^2 + 6xy + 9y^2$.

(*d*) $x^2 - y^2 = (x - y)(x + y)$ since $(x - y)(x + y)$ is $x^2 - y^2$.

Exercise

A

Give the simplest factors of each of the following expressions:

1. $6xy$, **2.** $5x^2$, **3.** $-35ab$,

4. $7x^2y$, **5.** $13xyz$, **6.** $-3m^2n^3$.

7. Multiply (*a*) $3(x + y)$, (*b*) $x(a + b)$, (*c*) $2x(a - b + c)$.

8. Factor (*a*) $5a + 5b$, (*b*) $am + bm$, (*c*) $3x + 3y - 3z$.

9. Multiply (*a*) $(x + 3)(x + 2)$, (*b*) $(y + 5)(y + 7)$.

10. Factor (*a*) $y^2 + 5y + 6$, (*b*) $x^2 + 11x + 28$.

11. Multiply (*a*) $(y - 4)(y - 2)$, (*b*) $(a - 3)(a - 5)$.

12. Factor (*a*) $m^2 - 6m + 8$, (*b*) $y^2 - 7y + 12$.

13. Expand (*a*) $(a + b)^2$, (*b*) $(x + 5)^2$.

14. Factor (*a*) $x^2 + 2xy + y^2$, (*b*) $y^2 + 6y + 9$.

15. Expand (*a*) $(x - y)^2$, (*b*) $(y - 3)^2$.

16. Factor (*a*) $a^2 - 2ab + b^2$, (*b*) $b^2 - 8b + 16$.

17. Multiply (*a*) $(m - n)(m + n)$, (*b*) $(a - 3)(a + 3)$.

18. Factor (*a*) $a^2 - b^2$, (*b*) $x^2 - 16$.

B

Factor:

19. $a^2 + 2ab + b^2$, **20.** $x^2 + 3x + 2$,

21. $ax + ay + az$, **22.** $a^2 - b^2$,

23. $m^2 - 2mn + n^2$, **24.** $m^2 - n^2$,

25. $am + bm - cm$, **26.** $x^2 + 5x + 6$,

27. $x^2 + 2xy + y^2$, **28.** $x^2 - 3x + 2$,

29. $7ab + 7ac$, **30.** $p^2 - 9$,

31. $r^2 + 2rp + p^2$, **32.** $x^2 - 25$,

33. $253 \times 6 + 253 \times 5$, **34.** $578 \times 225 - 578 \times 97$,

35. $25^2 - 24^2$, **36.** $15^2 - 12^2$.

A further examination will reveal that factorable algebraic expressions of more than one term can be classified into types for purposes of factoring.

Common Factor

It will be noticed that the number 5 is a factor common to each term of $5x + 5y + 5z$. Therefore

$$5x + 5y + 5z = 5(x + y + z).$$

Example: Factor $25x^2 - 30xy + 15xy^2$.

Solution: Note that $5x$ is a factor common to each term. Therefore $5x$ is one factor. Use division to find the other factor.

$$\frac{25x^2 - 30xy + 15xy^2}{5x} = 5x - 6y + 3y^2.$$

$$\therefore 25x^2 - 30xy + 15xy^2 = 5x(5x - 6y + 3y^2).$$

To check, multiply the factors:

$$5x(5x - 6y + 3y^2) = 25x^2 - 30xy + 15xy^2.$$

Note: (i) *In the factors of expressions of more than one term, monomial factors such as* $3b$, $5x$ *etc. are usually considered as simple factors.*

(ii) *The largest possible monomial factor common to each term should be used.*

Special Case of Common Factor

Example 1: Factor: $mx + nx + my + ny$

Solution: $(mx + nx) + (my + ny)$
$= x(m + n) + y(m + n)$
$= (m + n)(x + y)$

Since x is common to each of the first two terms, and y to each of the last two terms, these may be grouped. x is then factored out of the first group while y is factored out of the second. We now see that we have a factor $(m + n)$ common to each group. Therefore if $(m + n)$ is divided out of each group, $(x + y)$ is the other factor.

Example 2: Factor: $kx + 2x - ky - 2y$

Solution: $(kx + 2x) - (ky + 2y)$
$= x(k + 2) - y(k + 2)$
$= (k + 2)(x - y)$

Exercise

A

Factor:

1. $5x + 5y$,
2. $3x - 3$,
3. $6x - 12$,
4. $ax - ay$,
5. $2a + 2b + 2c$,
6. $x^2 - xy$,
7. $n^2 + n$,
8. $\frac{1}{2}hx + \frac{1}{2}hy$,
9. $3cx + 3c$,
10. $5x^2 - 10$,
11. $ab + bc - b$,
12. $a^2 + ab - ac$,
13. $6ab - 9ac$,
14. $3x^2 - 5xy + 2x$,
15. $x^2 + xy$,
16. $x^3y + y^3x$,

17. $4b^2c - 2bc^2$,
18. $\pi R + \pi r$,
19. $\pi R^2 - \pi r^2$,
20. $8 - 4y$,
21. $2x^2 - 8x$,
22. $3a^2b - 5ab^2$,
23. $0{\cdot}22ax - 0{\cdot}22by$,
24. $0{\cdot}7854D^2 - 0{\cdot}7854d^2$,
25. $(a + b)x + (a + b)y$,
26. $(x - y)a + b(x - y)$,
27. $3(ab - 2) + x(ab - 2)$,
28. $5(a + b) + 5(x + y)$,

B

Factor:

29. $(a + b)^2 + 5(a + b)$,
30. $(a - b)^2 - 3(a - b)$,
31. $ab + cd + ac + bd$,
32. $xa - ay - bx + by$,
33. $2 - p + 2q - pq$,
34. $a^3 + a^2b^2 + ab + b^3$,
35. $xm + 2m + 4x + 8$,
36. $ax + bx - ay - by$,
37. $x^3 - x^2 - x + 1$,
38. $3x^3 + 6x^2 + yx + 2y$,
39. $m^3 + mn + nm^2 + n^2$,
40. $2xy + 4x - 6 - 3y$,
41. $mn - np - qm + qp$,
42. $3ab - 6 - 2x + abx$.

Trial Trinomials

Example 1: Factor $x^2 + 5x + 6$.

Solution: Since we know that the product of two binomial factors often gives us a trinomial, we assume that the factors of this expression are binomials. Since the first term of the expression is x^2, we shall assume that the first term of each of the factors is x. Since the terms of the expression are all positive and the last term is $+6$, the second terms of the binomial factors could be $+6$ and $+1$ or $+2$ and $+3$. Thus we set up tentatively the products

$$\begin{array}{ll} x + 6 & x + 3 \\ \underline{x + 1} & \underline{x + 2} \end{array}$$

By multiplication $(x + 6)(x + 1) = x^2 + 7x + 6$, and

$$(x + 3)(x + 2) = x^2 + 5x + 6.$$

$$\therefore x^2 + 5x + 6 = (x + 3)(x + 2).$$

Example 2: Factor $x^2 - x - 6$.

Solution: Set up trial factors which show the first and the last term.

$$\begin{array}{llll} x + 6 & x - 6 & x + 3 & x - 3 \\ \underline{x - 1} & \underline{x + 1} & \underline{x - 2} & \underline{x + 2} \end{array}$$

From this selection, the product of the last set only gives the correct middle term. Therefore,

$$x^2 - x - 6 = (x - 3)(x + 2).$$

Example 3: Factor $2x^2 - 6x - 20$

Solution: $2x^2 - 6x - 20$
$= 2(x^2 - 3x - 10)$
$= 2(x - 5)(x + 2)$.

$x - 10$	$x + 10$	$x - 2$	$x - 5$
$x + 1$	$x - 1$	$x + 5$	$x + 2$

Example 4: $6x^2 - x - 12$.

Solution: $6x^2 - x - 12$ may or may not be the product of two simple binomials. Let us assume that it is.

Note: (i) $6x^2$ may be expressed as the product of two factors by $(2x)$ $(3x)$ or $(6x)$ (x).

(ii) -12 may be expressed as the product of two factors in any one of the following ways: (-3) $(+4)$, $(+3)$ (-4), (-2) $(+6)$, (-6) $(+2)$, (-12) $(+1)$, $(+12)$ (-1).

(iii) Set up tentatively possible products such as:

$2x + 3$	$2x - 3$	$2x + 4$	$2x - 4$
$3x - 4$	$3x + 4$	$3x - 3$	$3x + 3$

By actual multiplication it is found that $6x^2 - x - 12 = (2x - 3)(3x + 4)$.

Exercise

A

Factor:

1. $x^2 + 4x + 3$,
2. $x^2 + 5x + 6$,
3. $x^2 + 7x + 10$,
4. $a^2 + 7a + 12$,
5. $y^2 + 8y + 12$,
6. $m^2 + 13m + 12$,
7. $x^2 - 4x + 3$,
8. $x^2 - 3x + 2$,
9. $x^2 - 5x + 6$,
10. $x^2 - 7x + 6$,
11. $a^2 - a - 6$,
12. $m^2 + m - 6$,
13. $y^2 - 5y - 6$,
14. $y^2 + 5y - 6$.

B

Factor and check each by oral multiplication:

15. $a^2 + 8a + 12$,
16. $x^2 - 7x + 12$,
17. $k^2 + k - 30$,
18. $p^2 + 14p + 45$,
19. $a^2 + 13a + 36$,
20. $x^2 + 9x + 20$,
21. $x^2 + 12x + 20$,
22. $a^2 + 7a + 6$,
23. $m^2 - 8m + 15$,
24. $a^2 + 11a + 18$,
25. $z^2 - 16z + 64$,
26. $c^2 + 10c + 9$,

27. $b^2 + 15b + 14$,
28. $4 + 8r + 12r^2$,
29. $n^2 - n$,
30. $t^2 + 6t - 16$,
31. $x^2 - 3x - 28$,
32. $a^2 - 8ba + a$,
33. $n^2 + n - 56$,
34. $n^2 - n - 90$,
35. $a^2 - a - 42$,
36. $b^2 - 5b - 24$,
37. $c^2 - 7c - 30$,
38. $c^2 + 17c + 30$,
39. $c^2 + 31c + 30$,
40. $a^2 + 6a + 9$,
41. $m^2 + 8m + 16$,
42. $m^2 + 2m$,
43. $a^2 + 3ab + 2b^2$,
44. $m^2 + 6mn + 8n^2$,
45. $x^2 - 5xy + 6y^2$,
46. $x^2 - 2xy - 63y^2$,
47. $x^2 - xy - 12y^2$,
48. $8 - 6x + x^2$,
49. $2x^2 + 8x + 6$,
50. $2x^2 + 10x + 12$,
51. $ax^2 + 7ax + 6a$,
52. $2x^3 + 18x^2 + 36x$,
53. $5y^2 - 5yz - 30z^2$,
54. $3z^2 + 6z - 24$,
55. $mr^2 - 5mr + 6m$,
56. $5n^2 - 55n + 150$,
57. $4x^2 + 12x + 9$,
58. $2a^2 + 5a + 2$,
59. $3b^2 + 8b - 3$,
60. $49 + 42y + 9y^2$,
61. $6x^2 - 13xy + 6y^2$,
62. $9y^2 - 30y + 25$,
63. $20x^2 - x - 30$,
64. $6x^4 + 7x^3 - 5x^2$.

Special Cases of Trinomials

(1) Complete Squares

Since $(a + b)^2 = a^2 + 2ab + b^2$, we note that the complete square of a binomial has three terms, two of which are themselves squares. Hence, if two terms of a trinomial are squares, we suspect that the trinomial may be a complete square.

Example 1: Factor $4x^2 + 12xy + 9y^2$.

Since the first and last terms are complete squares and the middle term is positive, we suspect it to be the square of $2x + 3y$. By squaring $2x + 3y$ we get $4x^2 + 12xy + 9y^2$.

$$\therefore 4x^2 + 12xy + 9y^2 = (2x + 3y)^2.$$

After some practice, it will be sufficient to check the middle term only.

Example 2: Factor $x^2 - 10xy + 16y^2$.

Since two terms are square in themselves, we suspect a complete square. From the first and last terms and the sign of the middle term, we use $x - 4y$ as a trial.

Square $x - 4y$ and obtain $x^2 - 8xy + 16y^2$ and find that the expression is *not* a complete square.

If we examine it as a trial trinomial, we find that

$$x^2 - 10xy + 16y^2 = (x - 2y)(x - 8y).$$

Example 3: Factor $4x^2 - 24xy + 36y^2$.

Solution: $4x^2 - 24xy + 36y^2 = 4(x^2 - 6xy + 9y^2)$
$= 4(x - 3y)^2$.

(2) **Difference of Squares**

Since $(x + y)(x - y) = x^2 - y^2$, then $x^2 - y^2 = (x - y)(x + y)$.
Similarly $a^2 - b^2 = (a + b)(a - b)$ and $m^2 - n^2 = (m - n)(m + n)$.

The difference of the squares of two numbers is the product of the sum and difference of the same two numbers.

Example 1: Factor $4x^2 - 9y^2$.

Solution: $4x^2 - 9y^2 = (2x)^2 - (3y)^2$
$= (2x - 3y)(2x + 3y)$.

Example 2: Factor $18a^2 - 32b^2$.

Solution: $18a^2 - 32b^2 = 2(9a^2 - 16b^2)$
$= 2\{(3a)^2 - (4b)^2\}$
$= 2(3a + 4b)(3a - 4b)$.

Exercise

A

1. State the product of

(*a*) $(x - y)(x + y)$, (*b*) $(x + 3)(x - 3)$, (*c*) $(3x - y)(3x + y)$,
(*d*) $(5a - 4b)(5a + 4b)$, (*e*) $(x - 1)(x + 1)$, (*f*) $4(a - 3)(a + 3)$.

2. Expand:

(*a*) $(a - b)^2$, (*b*) $(x + 3y)^2$, (*c*) $(2a + 3)^2$,
(*d*) $(3x - 2y)^2$, (*e*) $(3x - 5)^2$, (*f*) $(5a - 7b)^2$.

3. State the factors of

(*a*) $a^2 + 2ab + b^2$, (*b*) $x^2 - 2xy + y^2$, (*c*) $a^2 + 4a + 4$,
(*d*) $x^2 - 6x + 9$, (*e*) $m^2 - 10m + 25$, (*f*) $a^2 - 8a + 16$.

4. State the factors of

(*a*) $x^2 - y^2$, (*b*) $x^2 - 9$, (*c*) $x^2 - 25$,
(*d*) $4x^2 - 121$, (*e*) $9x^2 - 16$, (*f*) $a^2b^2 - c^2$.

B

Factor:

5. $x^2 + 16x + 64$, **6.** $x^2 - a^2$,

7. $x^2 + 12x + 36$, **8.** $x^2 - 256$,

9. $a^2 + 6ab + 9b^2$,
10. $4x^2 - 9y^2$,
11. $m^2 - 6mn + 9n^2$,
12. $16 - 25y^2$,
13. $a^2 - 20a + 100$,
14. $1 - x^2$,
15. $4x^2 - 4x + 1$,
16. $a^2 - \frac{1}{4}$,
17. $25q^2 - 70pq + 49p^2$,
18. $49q^2 - 81p^2$,
19. $16b^2 + 40bc + 25c^2$,
20. $16b^2 - 25c^2$,
21. $x^2 + 5xy + 4y^2$,
22. $4y^2 - 8x^2$,
23. $4r^2 + 12rs + 9s^2$,
24. $4r^2 - 9s^2t^2$,
25. $2a^2 + 4ab + 2b^2$,
26. $2a^2 - 2b^2$,
27. $3x^2 - 3y^2$,
28. $5a^2 - 10ab + 5b^2$,
29. $ax^2 - ay^2$,
30. $am^2 + 4am + 4a$,
31. $5 - 45n^2$,
32. $a^4 - 4a^2b^2$,

C

Factor:

33. $(x + 3)^2 - y^2$,
34. $(a + b)^2 + 2(a + b)c + c^2$,
35. $(x + y)^2 - z^2$,
36. $4a^2 - 9(a + b)^2$,

Find the value of:

37. $25^2 - 24^2$,
38. $50^2 - 49^2$,
39. $99^2 - 98^2$,
40. $3^2 + 4^2$.
41. $(50 + 1)^2$,
42. $(70 + 1)^2$,
43. 31^2,
44. 61^2,
45. $(40 - 1)^2$,
46. 49^2,
47. 88^2,
48. 77^2.

49. What is the quotient when $a^2 - 9b^2$ is divided by $a - 3b$?

50. What is the square root of $25a^2 + 80ab + 64b^2$? What is the expansion of $(-5a - 8b)^2$?

51. What would have to be added to each of the following expressions to make them perfect squares:
(*a*) $8a^2$, (*b*) $a^2 + b^2$, (*c*) $x^2 - 2xy$, (*d*) $16a^2 + 8a$, (*e*) $a^2 + 6a + 7$?

Applications of Factoring

Most operations with fractions, in arithmetic and algebra, depend upon the principle that:

The value of a fraction remains unchanged when its numerator and its denominator are multiplied or divided by the same factor, zero excepted.

Reduction of Fractions to Lowest Terms

Example 1: Reduce to lowest terms each of the following simple fractions:

$$(a)\ \frac{24}{36}, \qquad (b)\ \frac{15x^2}{25xy}, \qquad (c)\ \frac{x^2-y^2}{3x^2+3xy}.$$

Solution: In each case, express each of the terms as the product of its simplest factors and divide the numerator and denominator by all the factors common to both terms.

$$(a)\ \frac{24}{36}=\frac{\overset{1}{\cancel{2}}\times\overset{1}{\cancel{2}}\times 2\times\overset{1}{\cancel{3}}}{\underset{1}{\cancel{2}}\times\underset{1}{\cancel{2}}\times\underset{1}{\cancel{3}}\times 3}=\frac{2}{3}.$$

Each term was divided by 12.

$$(b)\ \frac{15x^2}{25xy}=\frac{3\times\cancel{5}\times\cancel{x}\times x}{\cancel{5}\times 5\times\cancel{x}\times y}=\frac{3x}{5y}.$$

Each term was divided by $5x$.

$$(c)\ \frac{x^2-y^2}{3x^2+3xy}=\frac{(x-y)\cancel{(x+y)}}{3x\cancel{(x+y)}}=\frac{x-y}{3x}.$$

Each term was divided by $x+y$.

Example 2: Reduce to lowest terms each of the following compound fractions:

$$(a)\ \frac{14}{15}\times\frac{10}{77}, \qquad (b)\ \frac{3a}{5b}\times\frac{10bc}{6a^2}, \qquad (c)\ \frac{x^2+3x+2}{x^2-9}\times\frac{x^2-4x+3}{x^2+x-2}.$$

Solution: (a) $$\frac{14}{15}\times\frac{10}{77}=\frac{2\times\cancel{7}}{3\times\cancel{5}}\times\frac{2\times\cancel{5}}{\cancel{7}\times 11}=\frac{4}{33}.$$

$$(b)\ \frac{3a}{5b}\times\frac{10bc}{6a^2}=\frac{\cancel{3}\times\cancel{a}}{\cancel{5}\times\cancel{b}}\times\frac{\cancel{2}\times\cancel{5}\times\cancel{b}\times c}{\cancel{2}\times\cancel{3}\times\cancel{a}\times a}=\frac{c}{a}.$$

$$(c)\ \frac{x^2+3x+2}{x^2-9}\times\frac{x^2-4x+3}{x^2+x-2}=\frac{\cancel{(x+2)}(x+1)}{\cancel{(x-3)}(x+3)}\times\frac{\cancel{(x-3)}\cancel{(x-1)}}{\cancel{(x+2)}\cancel{(x-1)}}$$

$$= \frac{x+1}{x+3}.$$

Example 3: Divide $\frac{3x^2 - 6x}{x^2 + 3x + 2}$ by $\frac{5xy - 30y}{x^2 + x - 2}$.

Solution: The expression $= \frac{3x^2 - 6x}{x^2 + 3x + 2} \times \frac{x^2 + x - 2}{5xy - 10y}$

$$= \frac{3x(x-2)}{(x+2)(x+1)} \times \frac{(x+2)(x-1)}{5y(x-2)}$$

$$= \frac{3x(x-1)}{5y(x+1)}.$$

In arithmetic, the terms can often be factored at sight and the reduction made. This is not generally true of algebraic fractions. The student is advised to give undivided attention to the process of factoring the terms.

Exercise

A

Reduce to lowest terms:

1. $\frac{24}{8a}$, **2.** $\frac{6a}{12b}$, **3.** $\frac{x^2}{x}$, **4.** $\frac{3x}{3x^2}$,

5. $\frac{12ab}{36bc}$, **6.** $\frac{r^3}{r^2}$, **7.** $\frac{a^2}{a^6}$, **8.** $\frac{-6ab}{-18a}$,

9. $\frac{3}{3a + 3b}$, **10.** $\frac{ab}{ax + ay}$, **11.** $\frac{2x + 2y}{3x + 3y}$, **12.** $\frac{2x + 2y}{2x - 2y}$.

B

Reduce to lowest terms:

13. $\frac{6a^2b^2}{9abc^2}$, **14.** $\frac{km}{k^2m^3}$, **15.** $\frac{2xy^2}{6x^2y}$, **16.** $\frac{-10}{5ab}$,

17. $\frac{17 - 11}{17 + 11}$, **18.** $\frac{6a - 2a}{6a + 2a}$, **19.** $\frac{5^2 + 4^2}{5^2 - 4^2}$, **20.** $\frac{x^2 + xy}{xy}$,

21. $\frac{5a + 5b}{10a + 10b}$, **22.** $\frac{2a - 4b}{2a + 4b}$, **23.** $\frac{x^2 + x}{x^2 - x}$, **24.** $\frac{xy - xz}{wy - wz}$,

25. $\frac{a^2}{a^2 - a}$, **26.** $\frac{m + n}{m^2 - n^2}$, **27.** $\frac{m + n}{(m + n)^2}$, **28.** $\frac{m^2 - 9}{m^2 - 3m}$,

29. $\frac{a^2 + 2ab + b^2}{a^2 - b^2}$, **30.** $\frac{a^2 + 10a + 25}{a^2 + 8a + 15}$, **31.** $\frac{x^2 - 5x + 6}{x^2 + x - 6}$,

32. $\frac{ax + xy}{y^2 + ay}$, **33.** $\frac{a^2 - a - 6}{a^2 - 9}$, **34.** $\frac{a - b}{b - a}$, **35.** $\frac{x + y}{y + x}$,

36. $\dfrac{x^2-16}{x-4}$, **37.** $\dfrac{x^2-16}{4-x}$, **38.** $\dfrac{xy-4y}{x^2+4x-32}$,

39. $\dfrac{-a-b}{a+b}$.

Simplify the following:

40. $\dfrac{a}{b}\times\dfrac{b}{c}\times\dfrac{c}{a}$, **41.** $\dfrac{2ab}{6xy}\times\dfrac{3x^2}{5b^2}$,

42. $\dfrac{7x^3y^4}{21ax^4}\times\dfrac{15a^3b^5}{30b^4y^5}$, **43.** $\dfrac{2a}{a^2+ab}\times\dfrac{a^2-b^2}{3a-3b}$,

44. $\dfrac{x^2-25}{x^2-x-20}\times\dfrac{x^2+x-12}{x^2+2x-15}$, **45.** $\dfrac{x^2-1}{x^2-x}\times\dfrac{x}{2}$,

46. $\dfrac{x^2-3x}{x^2-9}\div\dfrac{x^2}{5x+15}$, **47.** $\dfrac{a^2+a-2}{a^2-a-6}\div\dfrac{a-2}{a-3}$,

48. $\dfrac{5a}{a-b}\div\dfrac{6a}{b-a}$, **49.** $\dfrac{x^2-xy}{xy+y^2}\div\dfrac{xy-y^2}{x^2+xy}$,

50. $\dfrac{a^2+2ab+b^2}{a^2+2ab}\times\dfrac{a}{a^2-b^2}$, **51.** $\dfrac{x^2+2x+1}{x^3}\times\dfrac{x}{x^2-1}$.

Addition and Subtraction of Algebraic Fractions

In arithmetic we learned that to express the sum of two or more fractions as a single equivalent fraction it was necessary to convert all of the individual fractions to equivalent fractions with equal denominators and then add the numerators.

Example 1: Combine $\dfrac{x}{4}+\dfrac{x}{2}-\dfrac{x}{3}$ into a single fraction.

Solution: A suitable common denominator is the lowest common multiple of 4, 2 and 3. The L.C.M. of 4, 2 and 3 is 12.

$$\text{Therefore } \frac{x}{4}+\frac{x}{2}-\frac{x}{3}=\frac{3x}{3(4)}+\frac{6x}{6(2)}-\frac{4x}{4(3)}$$

$$=\frac{3x+6x-4x}{12}$$

$$=\frac{5x}{12}.$$

Example 2: Add $\dfrac{5}{3x^2y}+\dfrac{6}{xy^2}$.

Solution: Factor each denominator completely: $3x^2y = 3\,(x)\,(x)\,(y)$,

$xy^2 = x\,(y)\,(y)$.

The L.C.M. of $3x^2y$ and xy^2 is $3x\,(x)\,(y)\,(y) = 3x^2y^2$. Therefore $3x^2y^2$ is a suitable common denominator and

$$\frac{5}{3x^2y} + \frac{6}{xy^2} = \frac{5y}{3x^2y^2} + \frac{18x}{3x^2y^2}$$

$$= \frac{5y + 18x}{3x^2y^2}.$$

Exercise

A

Add:

1. $\frac{4}{3x} + \frac{2}{3x}$,

2. $\frac{1}{x} + \frac{3}{2x} + \frac{2}{x}$,

3. $\frac{2x}{3} + \frac{3x}{4} + \frac{x}{6}$,

4. $\frac{1}{5} + \frac{2}{x}$,

5. $4 + \frac{3}{m} + \frac{1}{2m}$,

6. $\frac{x+2}{3} + \frac{x}{2}$,

7. $\frac{d-1}{d} + \frac{2}{3}$,

8. $\frac{4x}{3} + \frac{2x-1}{2}$,

9. $3 + \frac{2n-8}{3n}$,

10. $\frac{d^2-4}{d} + \frac{d+7}{3}$.

Subtract:

11. $\frac{2x+1}{2x} - \frac{4x}{3x}$,

12. $\frac{x+6}{2} - 2$,

13. $\frac{8}{x} - \frac{10}{3x} - \frac{7}{3}$,

14. $3 - \frac{3}{x}$,

15. $\frac{6x^2 + x}{2x} - \frac{3}{4}$,

16. $\frac{2}{xy} - \frac{4}{x^2y^2}$,

17. $\frac{x+4}{4x} - \frac{6}{14x^2}$,

18. $\frac{x}{6} - \frac{x+2}{9}$,

19. $\frac{pq-p}{5p} - \frac{q}{6}$,

20. $\frac{n-5}{4} - 1 - \frac{2n}{3}$.

Combine:

21. $\frac{2x+1}{3} - \frac{x}{4} + \frac{5x-2}{15}$,

22. $\frac{5-3x}{3x} + \frac{6x}{x^2} - \frac{7}{6x}$,

23. $\frac{x}{9} + \frac{2x}{15} - \frac{7x}{10}$,

24. $\frac{4x}{5} - \frac{7x}{6} + \frac{8x}{15}$.

The procedure for adding or subtracting fractions with binomial or trinomial terms is essentially the same as that for simpler fractions.

Example 1: Add $\frac{x-3}{x^2-4} + \frac{2x+3}{x^2-x-6}$.

Solution: Factor each of the denominators completely and obtain:

$$x^2 - 4 = (x-2)(x+2),$$
$$x^2 - x - 6 = (x+2)(x-3).$$

Therefore the Lowest Common Multiple of the denominators is $(x-2)(x+2)(x-3)$.

Convert each fraction to an equivalent fraction with $(x-2)(x+2)(x-3)$ as denominator.

Then
$$\frac{x-3}{x^2-4} + \frac{2x+3}{x^2-x-6} = \frac{(x-3)(x-3)}{(x^2-4)(x-3)} + \frac{(2x+3)(x-2)}{(x^2-x-6)(x-2)}$$
$$= \frac{(x^2-6x+9)+(2x^2-x-6)}{(x-2)(x+2)(x-3)}$$
$$= \frac{x^2-6x+9+2x^2-x-6}{(x-2)(x+2)(x-3)}$$
$$= \frac{3x^2-7x+3}{x^3-3x^2-4x+12}.$$

Example 2: Subtract $\frac{2x+1}{x^2-3x+2} - \frac{x-3}{2x^2-3x-2}$.

Factor each denominator: $x^2 - 3x + 2 = (x-2)(x-1)$,
$2x^2 - 3x - 2 = (2x+1)(x-2)$.

The L.C.M. of the denominators is $(2x+1)(x-2)(x-1)$.

Convert each fraction to an equivalent fraction with $(2x+1)(x-2)(x-1)$ for denominator.

Then
$$\frac{2x+1}{x^2-3x+2} - \frac{x-3}{2x^2-3x-2}$$
$$= \frac{(2x+1)(2x+1)}{(2x+1)(x-2)(x-1)} - \frac{(x-3)(x-1)}{(2x+1)(x-2)(x-1)}$$
$$= \frac{(4x^2+4x+1)-(x^2-4x+3)}{(2x+1)(x-2)(x-1)}$$
$$= \frac{4x^2+4x+1-x^2+4x-3}{(2x+1)(x-2)(x-1)}$$
$$= \frac{3x^2+8x-2}{2x^3-5x^2+x+2}.$$

Exercise B

Perform the indicated operations and simplify your results where possible:

1. $\frac{2}{x-4}+\frac{1}{x}-\frac{1}{2}$,
2. $\frac{4}{3x}-\frac{x+2}{x^2-2x}$,
3. $\frac{5-2x}{x^2}+\frac{4x-1}{x}-1$,
4. $\frac{x+12}{x^2-16}-\frac{3}{x-4}$,
5. $\frac{d}{d+3}-\frac{4}{d-3}$,
6. $\frac{x+2}{x^2-3x+2}-\frac{x-2}{x^2-x-2}-\frac{x-1}{x^2-1}$,
7. $\frac{5x}{2x-6}-2x-6$,
8. $\frac{7}{2x^2+6x}-3+\frac{3x-5}{x^2-9}$,
9. $\frac{2}{x^2+5x-14}-\frac{4}{x^2+14x+49}$,
10. $\frac{x}{x^3-x^2-x+1}+\frac{5}{x^2-2x+1}$,
11. $\frac{x+2}{x-2}-\frac{x-2}{x+2}$,
12. $\frac{3x}{x^2+6x+9}+\frac{x}{x^2-9}-\frac{x}{x^2+3x}$,
13. $\frac{4a+6}{2a+3}-\frac{1}{2}$,
14. $\frac{2x}{y(x+y)}-\frac{3y}{(x+y)^2}$,
15. $\frac{c}{c^3+2c}+\frac{4}{2c^2+4}-1$,
16. $\frac{2x+3}{6x^2+5x-6}+\frac{x+1}{9x^2+6x-8}$.

Other Applications of Factoring

To Collect Coefficients of Like Terms in an Expression

Example 1: Collect the coefficients of x and y in the expression

$$ax - by - cx - dy.$$

Solution:

$$\begin{aligned} & ax - by - cx - dy \\ &= ax - cx - by - dy \\ &= (ax - cx) - (by + dy) \\ &= (a - c)x - (b + d)y. \end{aligned}$$

Insert brackets with any necessary sign adjustment to combine like terms.

Example 2: Arrange in descending power of x, the expression

$$3ax - 8bx^2 - acx + ax^2 - 3b - c.$$

Solution:

$$\begin{aligned} & 3ax - 8bx^2 - acx + ax^2 - 3b - c \\ &= (ax^2 - 8bx^2) + (3ax - acx) - (3b + c) \\ &= (a - 8b)x^2 + a(3 - c)x - (3b + c). \end{aligned}$$

Example 3: Solve $ax + b = cx + d$ for x.

Solution: $ax + b = cx + d$

$ax - cx = d - b$

$(a - c)x = d - b$ Collect the coefficient of the unknown.

$x = \dfrac{d - b}{a - c}$ Divide each side by the coefficient of the unknown.

Simplifying Arithmetical Expressions

In some cases, algebraic products or factors can be used to advantage in simplifying arithmetical expressions.

Example 1: Simplify $325 \times 253 + 325 \times 147$.

Solution: $325 \times 253 + 325 \times 147 = 325(253 + 147)$

$= 325 \times 400$

$= 1{,}300.$

Example 2: Find the square of 65.

Solution: $65^2 = (60 + 5)^2$

$= 60^2 + 2 \times 60 \times 5 + 5^2$

$= 3{,}600 + 600 + 25$

$= 4{,}225.$

Example 3: Simplify 53×57 and 53×47.

Solution: $53 \times 57 = (50 + 3)(50 + 7)$

$= 50^2 + 10 \times 50 + 21$

$= 3{,}021.$

$53 \times 47 = (50 + 3)(50 - 3)$

$= 2{,}500 - 9$

$= 2{,}491.$

Example 4: Find the value of (*a*) $527^2 - 427^2$, (*b*) $\sqrt{37^2 - 12^2}$.

Solution: (*a*) $527^2 - 427^2$

$= (527 - 427)(527 + 427)$

$= 100 \times 954$

$= 95{,}400.$

(*b*) $\sqrt{37^2 - 12^2}$

$= \sqrt{(37 - 12)(37 + 12)}$

$= \sqrt{49 \times 25}$

$= 7 \times 5$

$= 35.$

Exercise

A

1. In the following expressions state the coefficient of x:
(*a*) $ax + bx$, (*b*) $cx - dx$, (*c*) $ax + b + cx$.

2. State the value of 10^2, 20^2, 14^2, 25^2, 35^2, 65^2, 85^2.

3. State the value of $10^2 - 9^2$; $30^2 - 29^2$; $35^2 - 25^2$; $48^2 - 47^2$.

B

Re-write the following expressions and collect the coefficients of x:

4. $bx - c - ax$,

5. $m - px - qx$,

6. $ax - b - cx - d$,

7. $abx + 3a - bcx + 5ab$.

Arrange the following in descending powers of x:

8. $ax^2 + bx^2 + cx + d$, **9.** $mx^2 + nx - 5x + c - 3$,

10. $ax^3 + bx^2 + cx + dx^3 - fx - gx^2$,

11. $-ax^2 - bx + cx + fx^2 + g$.

Solve the following equations for y:

12. $ay - b = 0$. **13.** $my + ny = p$.

14. $ay = 5y + c$. **15.** $ay + by + 3y - d = 0$.

Solve the following equations for x:

16. $2nx + 2x = p$. **17.** $x + nx = k$.

18. $\dfrac{x+1}{x+2} = \dfrac{a}{b}$. **19.** $\dfrac{x-a}{x-b} = \dfrac{c}{d}$.

20. $k = \dfrac{w(x-1)}{x}$. **21.** $A = \frac{1}{2}xy + \frac{1}{2}xz$.

Evaluate:

22. 25^2, **23.** 35^2, **24.** 45^2, **25.** 55^2,

26. 27×23, **27.** 36×34, **28.** 92×98,

29. 43×37, **30.** 29×31, **31.** 68×72,

32. 88×92, **33.** 89×91, **34.** 45×55.

Find the value of

35. $9^2 - 7^2$, **36.** $52^2 - 51^2$, **37.** $57^2 - 47^2$,

38. $\sqrt{41^2 - 40^2}$, **39.** $\sqrt{25^2 - 24^2}$, **40.** $\sqrt{26^2 - 24^2}$,

41. $\sqrt{17^2 - 8^2}$, **42.** $\sqrt{61^2 - 60^2}$, **43.** $\sqrt{5^2 - 4^2}$.

Review Exercise

A

1. State the products of

(*a*) $3(x - 6y)$, (*b*) $a(x - 3y)$, (*c*) $2x(x - 1)$,
(*d*) $(x + 3)(x + 4)$, (*e*) $(x + 7)(x + 8)$, (*f*) $(x + 3)(x + 1)$,
(*g*) $(x - 1)(x - 4)$, (*h*) $(x - 6)(x - 9)$, (*i*) $(x - 11)(x - 8)$,
(*j*) $(x + 6)(x - 5)$, (*k*) $(x - 6)(x + 3)$, (*l*) $(x - 4)(x + 4)$.

2. State the squares of

(*a*) $5ab$, (*b*) $2a^2b$, (*c*) $a + 6$,
(*d*) $3x - 4y$, (*e*) $5x^2 - y$, (*f*) $a^2 + b^2$.

B

Write the factors of each of the following expressions:

3. $a^2 + 5a + 6$, **4.** $y^2 - 3y - 10$,

5. $z^2 + 3z - 8$,
6. $a^2 + 12 + 36$,
7. $t^2 + 14t + 45$,
8. $a^2 - 5a - 50$,
9. $a^2 - 64b^2$,
10. $k^2 - k - 110$,
11. $c^2 + 16cd + 64d^2$,
12. $x^2 - \frac{1}{100}$,
13. $\frac{1}{4} - a^2$,
14. $1 - 2n - 24n^2$,
15. $1 - 12x + 35x^2$,
16. $c^2 + 7c + 10$,
17. $20y^2 - y - 1$,
18. $9a^2 - 81b^2$,
19. $3 - 27n^2$,
20. $4a^2 - 100$,
21. $4a^2 + 16$,
22. $5a^2 - 50$,
23. $3x^2 - 6x + 3$,
24. $ax^2 - ay^2$,
25. $\frac{1}{2}hx + \frac{1}{2}hy$,
26. $\pi R^2 - \pi r^2$,
27. $\pi rh + 2\pi r^2$,
28. $2{\cdot}4a^2 - 4{\cdot}8a$,
29. $a^4 - b^4$,
30. $a^4 - 1$,
31. $a^4 - 16$,
32. $a^4 - 81$.

C

33. What value of n will make $x + 3$ a factor of $x^2 + 7x + n$?

34. What value of n will make $x - 3$ a factor of $x^2 - nx + 15$?

35. What must be added to $x^2 - 8x$ to make it a perfect square?

36. What is the quotient when $x^2 - x - 12$ is divided by $x - 4$?

37. State the two square roots of
(*a*) 25, (*b*) $16a^2b^2$, (*c*) $16a^2 + 8a + 1$.

Factor:

38. $a(x + y) + b(x + y)$,
39. $a(x + y) + (x + y)$,
40. $(p - q)(a + b) + (p - q)c$,
41. $2 - 2(a - b)^2$,
42. $a^2 - (b + c)^2$,
43. $9 - (a - b)^2$,
44. $x^2 - (a - b)^2$,
45. $16 - (x - y)^2$.

46. Perform the indicated operations:

(*a*) $\dfrac{2}{x} - \dfrac{3}{y} - \dfrac{2}{x - y}$,

(*b*) $\dfrac{3(x - 6)}{x + 6} + \dfrac{2x}{x^2 + 12x + 36}$,

(*c*) $\dfrac{m}{m - n} + \dfrac{n}{n + m} - \dfrac{mn}{m^2 - n^2}$,

(*d*) $\dfrac{2x}{3x^2 + 9x} - \dfrac{2x^2 + 3x - 9}{4x^2 - 9}$.

47. The length of the three sides of a triangle are 5 units, $\dfrac{2x}{x - 1}$ units, and $\dfrac{x}{x + 1}$ units respectively. Find the perimeter of the triangle.

Express each of the following as a product of its simple factors:

48. $30 - 6p - pq + 5q$,

49. $n(a - b) - 3(b - a)$,

50. $21cd - 54c^2 + 20d^2$,

51. $9ax - a^3x$,

52. $(a + b)^2 + 2(a + b) + 1$,

53. $2x^3 + x^2 - 8x - 4$,

54. $x^2 - 6x + 9 - y^2$.

Simplify:

55. $\dfrac{4 - x^2}{m^2 - n^2} \div \dfrac{2 - x}{m + n}$,

56. $\dfrac{5x^2 + 10xy}{4r^2 - 36s^2} \cdot \dfrac{2rx - 4ry - 6sx + 12sy}{x^2 - 4y^2}$,

57. $\dfrac{2x^2 - 5xy + 3y^2}{18x^2 + 3x - 1} \cdot \dfrac{k^2 - 36x^2k^2}{2x^2 - 7xy + 3y^2} \div \dfrac{2xk + yk}{6x^2 + 5x + 1}$.

CHAPTER 15

Equations, Formulas and Problems

An example will serve to recall some of the basic principles used to solve equations.

What value of x will make the measure $3x + 5$ equal to the measure $5x - 4$?

It is given that	$5x - 4 = 3x + 5.$	(1)
Add 4 to each side	$5x = 3x + 5 + 4.$	(2)
Subtract $3x$ from each side	$5x - 3x = 5 + 4.$	(3)
Simplify each side	$2x = 9.$	(4)
Divide each side by 2	$x = 4\frac{1}{2}.$	(5)

Transposition of Terms

By the application of the addition and subtraction axioms, we have obtained equation (3) in which the unknown terms are on one side and the known terms on the other. The terms may then be combined to form the simplified equation (4).

The application of the addition and subtraction axioms can be accomplished more directly by *transposing* terms according to the following principle:

A term may be moved from one side of an equation to the other side if the sign of operation preceding it is changed.

Example 1: Solve $2x + 5 = 5x - 7$ for x.

Solution:	$2x + 5 = 5x - 7.$
Transpose the term $5x$ and the term 5	$2x - 5x = -7 - 5.$
Simplify each side	$-3x = -12.$
Divide each side by -3	$x = +4.$

Example 2: Solve $0{\cdot}03x - 0{\cdot}38 = 0{\cdot}43x + 0{\cdot}3.$

Solution:	$0{\cdot}03x - 0{\cdot}38 = 0{\cdot}43x + 0{\cdot}3.$
Transposing we get	$0{\cdot}03x - 0{\cdot}43x = 0{\cdot}3 + 0{\cdot}38.$
	$-0{\cdot}40x = 0{\cdot}68.$
Divide each side by $-0{\cdot}40$	$x = -1{\cdot}7.$

Verification: When $x = -1{\cdot}7$, the measure $0{\cdot}03x - 0{\cdot}38$
$= 0{\cdot}03(-1{\cdot}7) - 0{\cdot}38$
$= -0{\cdot}431.$

When $x = -1{\cdot}7$, the measure $0{\cdot}43x + 0{\cdot}3$
$= 0{\cdot}43(-1{\cdot}7) + 0{\cdot}3$
$= -0{\cdot}431.$

Since $0{\cdot}03x - 0{\cdot}38$ and $0{\cdot}43x + 0{\cdot}3$ have the same value when $x = -1{\cdot}7$, therefore $x = -1{\cdot}7$ is the correct solution.

Example 3: If the lengths of the sides of a triangle in linear units are represented by a, b and c respectively and the perimeter in the same linear units by p then $p = a + b + c$. Solve this formula for (i) a, (ii) b.

Solution: (i) $p = a + b + c.$

Transpose the terms b and c $p - b - c = a.$

Reverse the equation $a = p - b - c.$

(ii) $p = a + b + c.$

Reverse the equation $a + b + c = p.$

Transpose a and c $b = p - a - c.$

Exercise

A

1. Transpose the terms of the following equations so that the terms containing the unknown are on one side and the known terms on the other:

(a) $5x + 1 = 16.$ (b) $3x - 9 = 54.$
(c) $8x = 5x + 27.$ (d) $2x = 18 - 7x.$
(e) $7x - 8 = 6x + 18.$ (f) $4x + 1 = 2x - 8.$
(g) $3x - 7 = 8x - 12.$ (h) $x - 1 = 6 - x.$

2. Transpose and complete the solution orally:

(a) $2x + 8 = 24.$ (b) $5x - 7 = 8.$
(c) $3x - 7 = 2x + 8.$ (d) $5x - 1 = 3x + 13.$
(e) $2x - 5 = 8 - x.$ (f) $0{\cdot}3x - 0{\cdot}6 = 0{\cdot}2x.$

3. (a) State the two axioms that are included in transposition.
(b) State the two axioms that are not included in transposition.

4. In each of the following equations, express x in terms of y:

(a) $x + y = 5.$ (b) $x - y = 7.$
(c) $2x + y = 3.$ (d) $3x - 2y = 5.$

B

5. Solve and verify:

(a) $5x + 3 = 2x + 9.$ (b) $4x - 7 = 8x + 5.$
(c) $3x - 1 = 3 - 5x.$ (d) $18x + 4 = 10x - 20.$

6. Solve and verify:

(*a*) $0{\cdot}5x - 1 = 5 - 0{\cdot}3x$. (*b*) $0{\cdot}3 + 0{\cdot}5x = 0{\cdot}6 + 0{\cdot}7x$.

(*c*) $3y - 2{\cdot}1 = y + 5$. (*d*) $0{\cdot}01x + 0{\cdot}05 = 0{\cdot}3x - 0{\cdot}82$.

7. In each of the following equations, express y in terms of x:

(*a*) $3x + y = 18$. (*b*) $4y - 3x = 15$.

(*c*) $x - y = 7$. (*d*) $5x - 4y = 13$.

8. When $IR = E$, solve for I.

9. When $2s = a + b + c$, solve for a.

10. When $ay + b = 15$, solve for y.

11. When $ax = b - c$, solve for x.

12. When $mx - n - p = q$, solve for x.

C

13. When $ax + b = cx$, solve for x.

14. When $\frac{1}{2}hx + \frac{1}{2}hy = A$, solve for x.

15. If 4 pens cost \$2.40 less than 7 pens of the same quality, what is the price of each pen?

16. Solve: (*a*) $21 - 3x = 4x - 7$. (*b*) $35 - 4x = 5 + x$.

Equations Requiring Simplification

When the expressions used to form an equation are not in their simplest form, it is usually advisable to simplify them before proceeding with the actual solution.

Example 1: Divide \$100 among three persons so that the first will receive \$8 less than the second and the third will receive \$15 more than the first and second together.

Solution: Let the amount received by the second in dollars be x. Then the amount received by the first in dollars is $x - 8$.

The amount received by the third in dollars is $x + (x - 8) + 15$

$= 2x + 7$.

The total amount received in dollars is $x + (x - 8) + (2x + 7)$.

But the total amount is given in dollars as 100.

$$\therefore x + (x - 8) + (2x + 7) = 100$$
$$x + x - 8 + 2x + 7 = 100$$
$$4x - 1 = 100$$
$$4x = 101$$
$$x = 25{\cdot}25.$$

The first receives \$17.25.

The second receives \$25.25.
The third receives \$57.50.

Example 2: Solve and verify $6a - 2(4a + 3) = 4(2a - 1) + 3$.

Solution:

$$\begin{aligned} 6a - 2(4a + 3) &= 4(2a - 1) + 3 \\ 6a - (8a + 6) &= (8a - 4) + 3 \\ 6a - 8a - 6 &= 8a - 4 + 3 \\ -2a - 6 &= 8a - 1 \\ -2a - 8a &= -1 + 6 \\ -10a &= 5 \\ a &= -\tfrac{1}{2}. \end{aligned}$$

Verification: When $a = -\frac{1}{2}$,

$$\begin{aligned} &6a - 2(4a + 3) \\ &= 6(-\tfrac{1}{2}) - 2[4(-\tfrac{1}{2}) + 3] \\ &= -3 - 2(+1) \\ &= -5. \end{aligned}$$

When $a = -\frac{1}{2}$,

$$\begin{aligned} &4(2a - 1) + 3 \\ &= 4[2(-\tfrac{1}{2}) - 1] + 3 \\ &= 4(-2) + 3 \\ &= -5. \end{aligned}$$

Since $6a - 2(4a + 3)$ and $4(2a - 1) + 3$ have the same value when $a = -\frac{1}{2}$, therefore $a = -\frac{1}{2}$ is correct.

Example 3: Solve and verify $(x + 3)^2 - (x - 2)(x + 5) = 4$.

Solution:

$$\begin{aligned} (x + 3)^2 - (x - 2)(x + 5) &= 4 \\ (x^2 + 6x + 9) - (x^2 + 3x - 10) &= 4 \\ x^2 + 6x + 9 - x^2 - 3x + 10 &= 4 \\ 3x + 19 &= 4 \\ 3x &= 4 - 19 \\ 3x &= -15 \\ x &= -5. \end{aligned}$$

Verification: When $x = -5$,

$$\begin{aligned} &(x + 3)^2 - (x - 2)(x + 5) \\ &= (-5 + 3)^2 - (-5 - 2)(-5 + 5) \\ &= (-2)^2 - (-7)(0) \\ &= +4. \end{aligned}$$

Therefore $x = -5$ is the correct solution.

Exercise B

1. Solve and verify each of the following equations:

(*a*) $6(y - 2) - 42 = 0$. (*b*) $2(x - 3) + 5x = 21$.
(*c*) $6x - 2(4x - 2) = 18$. (*d*) $3(a + 5) = 2(a - 5) + 20$.

2. Solve and verify:

(*a*) $(x - 3)(x + 2) = x^2 + x$.
(*b*) $(x + 3)^2 + (x - 5)^2 = 2(x + 7)^2$.
(*c*) $3(a - 1)^2 = (3a - 1)(a + 1)$.

3. Solve:

(*a*) $12(x - 4) + 3(x + 16) = 0$. (*b*) $2(3x - 10) = 5(x + 9) - 64$.
(*c*) $15(y - 4) - 2(y - 9) + 5(y + 6) = 0$.
(*d*) $21(5x - 2) - 8(2x + 3) = 112$.

4. Solve:

(*a*) $(x + 1)(x - 3) - (x + 4)(x + 2) = 5$.
(*b*) $(2x - 3)^2 - (2x - 7)^2 = 200$.
(*c*) $2(x - 5)^2 = (2x - 6)(x - 6)$.

5. Solve:

(*a*) $\dfrac{3x - 6}{3} + \dfrac{5x - 10}{5} = 24$. (*b*) $\dfrac{2x - 8}{2} + \dfrac{7x - 14}{7} = 6$.

(*c*) $\dfrac{4x - 12}{4} - \dfrac{6x - 9}{3} = 8$. (*d*) $\dfrac{7x + 28}{7} + \dfrac{4x - 20}{4} = 3$.

6. Divide \$100 among John, Bill and George in such a way that John's share is \$8 more than Bill's and George's share is twice as much as John's.

7. Each of the equal sides of an isosceles triangle is 3 feet longer than the base. If its perimeter is 30 feet, what is the length of each side?

8. Find three consecutive numbers such that the sum of the first, twice the second and four times the third is 129.

9. The lengths of two squares are $x + 5$ and $x - 3$ feet respectively. If the difference of their areas is 208, find the length of each square.

10. I have 25 coins in nickels and dimes. If I have \$1.45, how many of each kind have I?

11. I have 60 coins in nickels, dimes and quarters. I have twice as many nickels as dimes. The value of the coins is \$8.95. Find the number of each.

12. A boy gets 3 marks for each question he gets right, and loses 2 marks for each question wrong. He does 24 questions and gets 47 marks. How many were right?

13. A car travels on the average 8 miles per hour faster than a bus. They meet going in opposite directions and 3 hours later they are 264 miles apart. How fast was each travelling?

14. A man invests \$10,000, part of which is in a mortgage at 6% and the balance in bonds at 4%. If the total interest is \$560, how much was invested in the mortgage?

15. A man invests \$5,000, a part of which is in preferred stock at 5% and the balance in government bonds at 3%. If the difference in the income of the two investments is \$170, how much was invested in each?

16. If 25 is subtracted from a certain number, the remainder is 9. Find the number.

17. If A is three times as old as B and the sum of their ages is 84, find the age of each.

18. The formula for the circumference of a circle is $c = \pi d$. Express d in terms of c and π.

19. The sum of three consecutive odd numbers is 27. Find the numbers.

20. If one-half of a number decreased by $\frac{1}{4}$ is the same as the original number diminished by $\frac{1}{12}$, what is the number?

21. A rectangle is three times as long as it is wide. If the perimeter is 128 feet, find the lengths of its sides.

22. What sum must be invested at $4\frac{1}{2}$% per annum to provide \$300 annually for a scholarship?

23. Solve $0{\cdot}7x + 0{\cdot}3 = 0{\cdot}4x + 1{\cdot}2$.

24. If a soft drink costs 2 cents more than an ice-cream cone and a boy buys 5 drinks and 5 cones at a total cost of 70 cents, what is the cost of a cone?

25. If a number is multiplied by 3·2, and 6·8 is added to the product, the sum is 32·72. Find the number.

26. A rope 52 feet long is to be marked in two places so that each of the end pieces will be 6 times as long as the middle piece. How far from each end should the marks be placed?

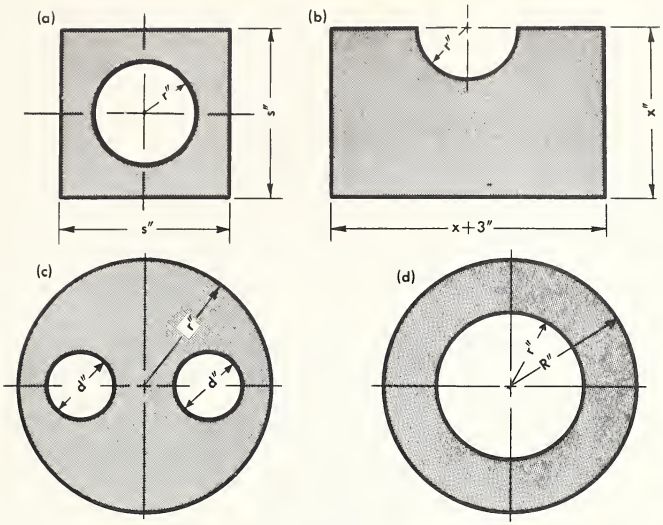
27. Is $x = 3\frac{2}{3}$ the solution for the equation $5x - 8 = 2x + 3$?

28. A rectangle $x + 6$ inches long and $x + 4$ inches wide is 75 square inches greater in area than a square $x + 3$ inches to a side. Find the dimensions of the rectangle.

29. In the formula $y = mx + b$, find the value of m if $y = 14{\cdot}8$, $x = 1{\cdot}2$ and $b = 0{\cdot}4$.

30. Solve and verify $3{\cdot}6x + 0{\cdot}17 = 1{\cdot}7x + 0{\cdot}664$.

31. Express as a formula the measure of the area of the shaded portion of each of the following figures:



32. (*a*) Given that $r = as + b$, solve for (i) b, (ii) s.

(*b*) Given that $a = \frac{h}{2}(x + y)$, solve for (i) h, (ii) y.

Equations with Fractional Coefficients

The relationships in problems often involve equations with fractional coefficients. Consider the following example:

A boy can row a boat upstream at 3 miles per hour and downstream at 5 miles per hour. How far upstream can he go and return in 2 hours?

Let the distance in miles be x.

Time to go in hours is $\frac{x}{3}$.

Time to return in hours is $\frac{x}{5}$.

Total time for the trip in hours is $\frac{x}{3} + \frac{x}{5}$.

But the time allowed in hours is 2.

$$\therefore \frac{x}{3} + \frac{x}{5} = 2.$$

To clear the equation of fractions, multiply each side by the least common multiple of the denominators.

$$15 \times \frac{x}{3} + 15 \times \frac{x}{5} = 15 \times 2$$
$$5x + 3x = 30$$
$$8x = 30$$
$$x = 3\tfrac{3}{4}.$$

The distance he can row upstream is $3\frac{3}{4}$ miles.

Example: Solve $\dfrac{a+3}{6} - \dfrac{4-a}{3} = \dfrac{7}{4}$.

Solution:

$$\frac{a+3}{6} - \frac{4-a}{3} = \frac{7}{4}$$

Multiply each term by 24.

$$24 \times \frac{a+3}{6} - 24 \times \frac{4-a}{3} = 24 \times \frac{7}{4}$$
$$4(a+3) - 8(4-a) = 6 \times 7$$
$$4a + 12 - 32 + 8a = 42$$
$$12a = 42 + 20$$
$$12a = 62$$
$$a = 5\tfrac{1}{6}.$$

Example: Solve $\frac{1}{2}(x+3) - \dfrac{2}{3}(x-2) = 2x$.

Solution:

Multiply each term by 6.

$$3(x+3) - 4(x-2) = 12x$$
$$3x + 9 - 4x + 8 = 12x$$
$$-13x = -17$$
$$x = 1\frac{4}{13}.$$

Example: Solve $\dfrac{3}{2}(\dfrac{2x}{5} - 1) - \frac{1}{4}(x+1) = \dfrac{1}{3}(2x - \frac{1}{2})$.

Solution:

Multiply each term by 60.

$$18(2x-5) - 15(x+1) = 10(4x-1)$$
$$36x - 90 - 15x - 15 = 40x - 10$$
$$21x - 105 = 40x - 10$$
$$-19x = 95$$
$$x = -5.$$

Cross Multiplication

Example: Solve $\dfrac{16}{9} = \dfrac{64}{x}$.

Solution:

$$\frac{16}{9} = \frac{64}{x} \tag{1}$$

$$9x \times \frac{16}{9} = \frac{64}{x} \times 9x \tag{2}$$

$$16x = 9 \times 64 \tag{3}$$

$$x = 36.$$

If we compare equation (3) with equation (1), we notice that the product of the numerator of one fraction and the denominator of the other is equal to the product of the denominator of the first fraction and the numerator of the second.

When two fractions are equal to each other, the equation may be simplified by equating the two products formed by cross multiplication.

Example: Solve $\frac{3}{4} = \frac{x}{9}$.

Solution:

$$\frac{3}{4} = \frac{x}{9}$$

By cross multiplication

$$4x = 27$$

$$x = 6\tfrac{3}{4}.$$

Exercise B

Solve and verify each of the following equations:

1. $x + \frac{x}{2} = 18.$ **2.** $\frac{1}{2}x - \frac{1}{6}x = 6.$ **3.** $\frac{3}{5}x - 4 = 4 - x.$

4. $\frac{2}{3}y - \frac{1}{2}y = 1.$ **5.** $\frac{6}{7}k + 2 = 11.$ **6.** $\frac{2}{3}x + \frac{3}{4}x = \frac{7}{8}x - 1\frac{1}{2}.$

7. $x = \frac{2}{7}x + 10.$ **8.** $n = 6 + \frac{1}{3}n.$ **9.** $\frac{2}{3}x - 8 = 0.$

10. $\frac{3}{4}m - 1 = 5.$ **11.** $\frac{3}{5}t + \frac{1}{2}t = \frac{33}{20}.$ **12.** $\frac{x}{2} + \frac{x}{3} = 15.$

13. $x + \frac{1}{2}x + \frac{x}{3} = 11.$ **14.** $\frac{x}{3} = 7 - \frac{x}{6}.$ **15.** $\frac{a}{4} + \frac{2a}{3} = 0.$

16. $\frac{m}{7} + m + \frac{1}{2} = 1\frac{9}{14}.$ **17.** $k - \frac{k}{2} - \frac{k}{3} = 2.$

18. $x - \frac{x}{5} = \frac{x}{10} + 6.$

Solve

19. $\frac{5x+4}{3} - \frac{3x+5}{2} = 0.$ **20.** $\frac{5}{7}(x-3) - \frac{2}{5}(x+1) = x - 19.$

21. $\dfrac{4x+3}{3} = \dfrac{x-3}{2}$.

22. $\frac{1}{5}(x+16) + 9 = 2x - \frac{1}{3}(4x-17)$.

23. $\frac{3}{4}(y-1) - \frac{2}{3}(y+1) = 1\frac{1}{2}$.

24. $\dfrac{k+1}{3} + \dfrac{5}{21} = \dfrac{4-k}{7}$.

25. $m - \dfrac{m-1}{2} = 1 - \dfrac{m-2}{3}$.

26. $\dfrac{x-7}{2} - \dfrac{x+3}{5} = 0$.

27. $\dfrac{y}{4} + \dfrac{2y+8}{3} - 3 = 0$.

28. $\frac{1}{5}(5+7p) = \frac{1}{7}(7+5p)$.

29. $\frac{1}{3}(5x-19) = \frac{1}{2}(3x-10) + 1$.

30. $\dfrac{a}{3} = 6 - 2(a-7)$.

31. $\dfrac{x}{3} = \dfrac{5}{8}$.

32. $\dfrac{3}{x} = \dfrac{7}{4}$.

33. $\dfrac{x}{64} = \dfrac{3}{7}$.

34. $\dfrac{y}{32} = \dfrac{5}{9}$.

35. $\dfrac{k}{16} = \dfrac{5}{7}$.

36. $\dfrac{3x}{4} = 9$.

37. $\dfrac{y+4}{2} = \dfrac{y+10}{3}$.

38. $\dfrac{2x-3}{5} = 3$.

39. $\dfrac{x}{0{\cdot}2} = 4$.

40. $\dfrac{5x+2}{7x+1} = \dfrac{3}{4}$.

41. $\dfrac{140n}{100} = 84$.

42. Solve for h in the formula $A = \frac{1}{2}bh$.

43. Solve for a in the formula $A = \frac{1}{2}h(a+b)$.

44. Solve for C in the formula $F = 32 + \frac{9}{5}C$.

45. The perimeter of a semicircle is given by $p = d + \frac{1}{2}\pi d$. If $\pi = 3{\cdot}14$ and $p = 12{\cdot}6$, solve for d.

46. If a pail can be filled by one hose in 10 minutes and by another hose in 4 minutes, then it can be filled by both hoses together in t minutes where t is the root of the equation $\dfrac{1}{t} = \dfrac{1}{10} + \dfrac{1}{4}$. Find t.

47. The total surface of a cylinder is given by the formula $A = 2\pi r^2 + 2\pi rh$. Solve for h.

48. In each of the following formulas solve for the letter indicated:

(*a*) When $d - \dfrac{1{\cdot}299}{n} = r$, solve for d;

(*b*) When $\dfrac{R}{r} = \dfrac{t}{T}$, solve for T;

(*c*) When $V = \frac{1}{3}ah$, solve for (i) a, (ii) h.

C

49. Solve for x, when $\frac{1}{72} + \frac{1}{x} = \frac{1}{12}$.

50. Find a number such that if you subtract 6 and divide by 3 you get the same as if you had added 6 to the number and divided by 5.

51. If $h = \frac{r(n-1)}{n+1}$ and $n = \frac{2}{3}$, find r in terms of h.

52. Solve $\frac{1}{3}(\frac{2}{5}x - 5) - \frac{1}{4}(\frac{1}{5}x - 1) = 3$.

53. Solve $\frac{2}{1-x} = \frac{3}{1-2x}$.

54. If $5x + 3y = 7$ solve for x. In order to make x negative, what is the least possible integral value for y?

55. Solve $\frac{m}{3} + 7 = \frac{1}{2}(2m - 4)$.

56. Solve $4x(-3 + 2x) = 2[3 - (1 - 4x^2)]$.

57. Solve $a - a(1 - a) + a(a - 1) = 2 - a(-1 - 2a)$.

58. If $L = a + (n - 1)d$ solve for d.

59. If $E = iR + ir$, solve for i and R.

Review Exercise

A

1. If the length of each of the four sides of a quadrilateral in linear units is a, b, c and d respectively, the perimeter is given by the formula $p = a + b + c + d$. Solve for b.

2. Solve and verify:

(*a*) $6x + 9 = 45$. (*b*) $7x - 13 = 6x - 7$.

(*c*) $5x + 3x = 2x + 72$. (*d*) $1 - 4x = 2 - 6x$.

3. What sum of money loaned at 6% will yield an annual income of $1,250?

4. In each of the following problems, express x in terms of the other measures given:

(*a*) $\frac{x}{a} = b$. (*b*) $x - b = c$.

(*c*) $ax = k$. (*d*) $x + 5 = w$.

(*e*) $ax + b = c$. (*f*) $\frac{1}{2}x + a = b$.

(*g*) $\frac{a}{x} = b$. (*h*) $ax + bx = c$.

B

5. Solve $5(a - 3) = 4(a - 4)$.

6. Solve and verify $\frac{1}{2}x - \frac{1}{3}x = 25$.

7. Solve $(x + 4)(x + 3) = (x + 1)(x - 1) - 1$.

8. Solve $\dfrac{x + 1}{5} - \dfrac{x - 1}{4} = 5$.

9. Solve and verify $\dfrac{(x - 1)(x + 2)}{4} + \dfrac{(x - 3)^2}{10} = \dfrac{7}{20}(x - 3)(x + 1) + \dfrac{2}{5}$.

10. Solve $\dfrac{18}{24} = \dfrac{36}{x}$.

11. Solve $\dfrac{2x + 1}{4} = \dfrac{x + 3}{3}$.

12. When $A = \frac{1}{2}h(x + y)$ solve for x in terms of the other measures.

13. When $V = \frac{1}{3}\pi r^2 h$ solve for h in terms of the other measures.

14. Find the value of x from the equation $\dfrac{2}{k} = \dfrac{4}{x}$.

15. Solve and verify $\frac{3}{4}x = 3 + \frac{5}{6}x$.

16. Solve and verify $(x + 4)^2 - (x - 3)^2 = 77$.

17. Divide an estate of $27,000 among three beneficiaries so that the first receives twelve times as much as the second, and the third receives five times as much as the second.

18. The sides of a rectangle in inches are expressed by the numbers $\frac{2}{5}x + 6$ and $\frac{1}{2}x$. If the perimeter is 48 inches, find the value of x.

19. Solve and verify:

(*a*) $(x - 3)^2 + (x + 1)^2 = 2x^2 - 18$.

(*b*) $(x - 1)(x + 2) + (x - 3)(x + 4) = 2x^2 - 12$.

(*c*) $\dfrac{x}{6} - \dfrac{x - 2}{10} = 3$.

(*d*) $\dfrac{x + 1}{5} + \dfrac{x - 1}{4} = \dfrac{2 - x}{3} + \dfrac{1}{15}$.

20. When $\dfrac{R}{r} = \dfrac{d}{D}$ solve for (i) d, (ii) D.

21. Solve for F in the formula $C = \frac{5}{9}(F + 32)$.

22. The side of one square is $x + 5$ inches and the side of another square is $x - 5$ inches. If the difference in their areas is 140 square inches, find the value of x.

23. If $c^2 - (x - a)^2 = b^2 - x^2$, solve for x.

24. If $\frac{x}{15{\cdot}5} = 0{\cdot}75$, find x.

25. Is $x = 0{\cdot}5$ the correct solution for the equation $2x^2 + x - 1 = 0$?

26. If $\frac{x+1}{2}$ exceeds $\frac{x-1}{4}$ by $\frac{13}{16}$, determine the value of x.

C

27. Solve $S = \frac{n}{p} + L$ for p.

28. Solve $Sx + Px = c$ for x.

29. Solve and verify:

(a) $\frac{1}{3}(x + 2) + \frac{1}{2}(2x - 1) = \frac{2}{3}$.

(b) $1.8m + 2.4 = 0.6m - 1.2$.

(c) $2 - \frac{2x - 3}{2x + 1} = \frac{x + 10}{x - 1}$.

(d) $\frac{1}{3}(\frac{x}{2} + 1) = \frac{2x}{3} + 8$.

(e) $\frac{x - 1}{3x + 2} = \frac{x^2 - 4}{3x^2 + 6x}$.

30. The resistance of an electric wire is 12 ohms. The wire is cut into two parts so that the resistance of one part is 2 ohms less than twice that of second part. Find the resistance of the two parts.

31. One gasoline tank of a truck holds 20 gallons more than the other tank. When the larger tank is $\frac{1}{4}$ full it has the same amount of gasoline as $\frac{3}{4}$ of the smaller tank. Find the capacity of each of the gasoline tanks.

32. The outside dimensions of a picture frame are x in. and $(2x - 9)$ in. respectively. If the frame is 3 in. wide,
(*a*) Write an expression for the area of the picture opening.
(*b*) Find the area of the picture opening if its perimeter is 30 in.

33. If $F = \frac{9}{5}C + 32$ gives the relationship between temperatures expressed in degrees Fahrenheit and in degrees Centigrade, find the change in the Centigrade reading due to an increase of 40° Fahrenheit.

CHAPTER 16

Graphs and Simultaneous Equations

Suppose that we drive along a highway easterly from an intersection point at a constant speed of 30 miles per hour. A continuous record of our position will include two variable measurements, time and distance. Since the speed is assumed to be constant, the distance depends upon the time. In this case, distance is the **dependent variable** and time is the **independent variable.** Relationships such as this may be expressed in any one of three different ways: (i) by a table of values, (ii) by an equation or formula, or (iii) by a graph.

Table of Values

A record of our changing position could be shown by a table as follows:

Time in hours.	1	2	3	4	5	6	7	8
Distance in miles. .	30	60	90	120	150	180	210	240

A table expresses a relationship for specific values of the variables only.

The Relationship Expressed by a Formula

A record of our changing position might be expressed by a formula developed as follows:

Let the variable distance from the starting point in miles be d.

Let the time in hours be t.

The rate in miles per hour is 30.

Then, the distance in miles is $30t$.

Then the record of our changing position is expressed by the formula

$$d = 30t.$$

For any value of t this formula gives the corresponding value of d. A formula expresses the relationship for all values of the variables.

The Relationship Expressed by a Graph

A record of our changing position might be expressed graphically.

Draw **OT**, a straight-line representation of the time in hours. Draw **OD** perpendicular to **OT**. Let **OD** be a straight-line representation of the distance

in miles. **OT** and **OD** are **axes of reference** of the graph and **O** is the **origin.**

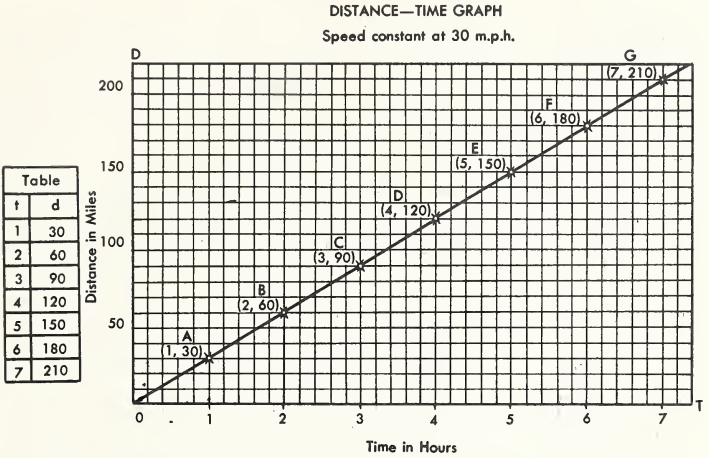


Table	
t	d
1	30
2	60
3	90
4	120
5	150
6	180
7	210

From the table of values for this relationship, it is found that when the time is 1 hour the distance is 30 miles. The point **A**, 1 unit to the right of **OD** and 30 units above **OT**, represents this set of values on the graph. In a similar way, plot other points on the graph. Join these points.

The continuous line expresses the relationship for all values of the variables within the range plotted. Since this graph expresses the same relationship as the formula $d = 30t$, it is the graph of $d = 30t$.

Each graph should have the axes of reference suitably scaled and properly labelled.

Exercise B

On squared paper, draw graphs of the relationships expressed by the following tables of values:

1. Temperature records made on a certain day were as follows:

Time of day	8 *a.m.*	9 *a.m.*	10 *a.m.*	11 *a.m.*	12 *noon*	1 *p.m.*	2 *p.m.*	3 *p.m.*	4 *p.m.*	5 *p.m.*
Temperature in degrees	20	25	28	35	40	45	37	36	36	35

(*a*) From your graph determine the approximate temperature at (i) 10:30 a.m., (ii) 2:30 p.m.

(*b*) At what time of the day was the temperature (i) 27°, (ii) 41°?

2. The heights of a large number of boys of the same age were measured and the results when tabulated were as follows:

Height in inches ...	60	61	62	63	64	65	66	67	68
Number of boys ...	7	33	110	210	275	208	105	36	8

Represent this data graphically. Graphs similar to this occur regularly in statistical work.

3. In an electrical experiment, the resistance in a circuit was maintained constant at 2 ohms, and records were kept of the amount of current required for different strengths of voltage. The following table indicates the results:

Voltage in volts	2	3	4	5	6	7	8	9	10	11	12	13
Current in amperes	1	1·5	2	2·5	3	3·5	4	4·5	5	5·5	6	6·5

Draw a graph to represent this relationship. From the graph determine approximately the current when the voltage is (i) 3·5 volts, (ii) 8·5 volts.

4. In an electrical experiment, the voltage was kept constant at 60 volts and the current strengths were measured when different resistances were used. Represent the following data graphically.

Resistance in ohms	2	3	4	5	6	8	10	12	15	16	20	24	30
Current in amperes	30	20	15	12	10	7·5	6	5	4	3·75	3	2·5	2

(*a*) What is the current when the resistance is (i) 4·5 ohms, (ii) 11 ohms?
(*b*) What is the resistance when the current is (i) 7 amp., (ii) 4·5 amp., (iii) 16 amp?

5. (*a*) State the formula for the circumference of a circle in terms of the radius.
(*b*) Prepare a table of values which will show the change in the circumference of the circle as the radius varies from 0 to 10.
(*c*) Construct a graph to illustrate this relationship.

6. The cost of an ice-cream cone is 6 cents.
(*a*) Prepare a table of values which will show the cost of any number of cones from 1 to 15.
(*b*) Construct a graph to show the relationship between the number of cones and their cost.
(*c*) If the cost is c cents and the number of cones is n, what is the relationship between c and n?

7. (*a*) Prepare a table of values to show the relationship between the area of a rectangle and its length, if the width is 3 inches and the length varies from 0 to 15 inches.

(*b*) Draw the graph of this relationship and read from it the following values:

(i) the area when the length is $5\frac{1}{2}$ inches;

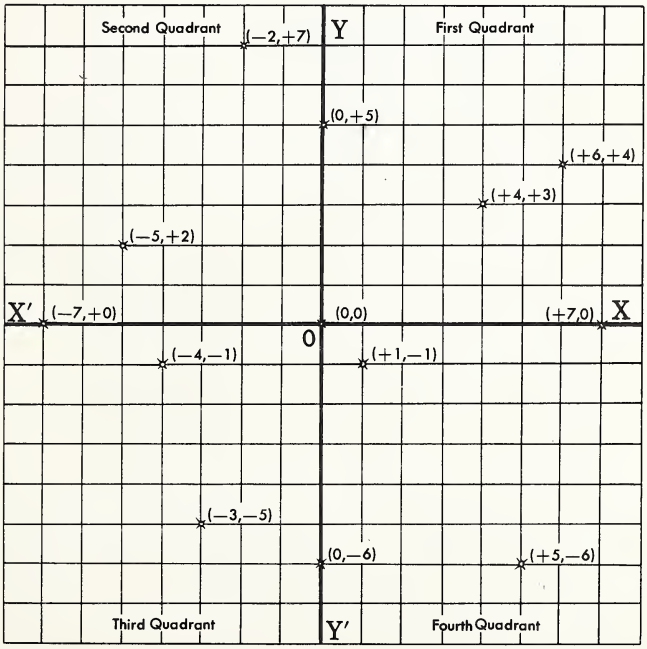
(ii) the length when the area is 38 square inches.

8. A rectangle with variable length and width has a constant area of 24 square inches. Draw a graph which shows the relationship between the length and the width for a range of values of the width from 1 to 24.

9. The volume of a gas varies inversely as the pressure according to the formula $V = \frac{18}{P}$. Draw a graph to show the relationship between the pressure and the volume for the range of values of P from 1 to 18.

Cartesian System of Co-ordinates

If the axes of reference are extended through the origin, negative as well as positive values of the variables may be plotted.



In the diagram shown, the **rectangular axes X′OX** and **YOY′** divide the plane into four parts called **quadrants.** Several points have been plotted. The

first number gives the measure of the distance to the right or left of **YOY′**, the **y-axis.** The second number gives the measure of the distance above or below **XOX′**, the **x-axis.** These numbers are the **co-ordinates** of the point.

In general, the co-ordinates of any point on the graph may be expressed as (x, y).

Exercise

B

1. Construct a system of rectangular axes and plot the positions of the following points: **A** (5, 6); **B** (4, 2); **C** (−3, 4); **D** (0, 6); **E** (−4, −3); **F** (−8, +7); **G** (−6, 0); **H** (6, −5); **J** (0, −4); **O** (0, 0).

2. The following information was obtained for two variables:

Values of x	8	2	−10	0	5	1	3	−8	−3	+9
Values of y	−5	1	+13	3	−2	2	0	+11	+6	−6

(*a*) Construct a graph of the relationship shown above.
(*b*) What is the value of y when (i) $x = 4$, (ii) $x = -4$?
(*c*) What is the value of x when (i) $y = 10$, (ii) $y = -4\frac{1}{2}$?

3. The relationship between the temperature readings on a Fahrenheit and Centigrade thermometer is given by the formula $F = \frac{9}{5}C + 32$.
(*a*) Make a table showing the values of F corresponding to the values of C for every 20 degrees for the range −80 to 100.
(*b*) Draw a graph of the relationship above.
(*c*) From the graph, read the values of C corresponding to the following values of F: 5, 23, −40, 59, 95, 135, 165.
(*d*) Read from the graph the values of F corresponding to the following values of C: −25, −5, 15, 75, 95.

4. (*a*) Draw a graph of the formula $A = s^2$ for the range of s between −6 and +6.
(*b*) Read from your graph the approximate values of s when A is (i) 16, (ii) 25, (iii) 20, (iv) 12, (v) 27.

5. (*a*) Draw a graph showing the relationship between the variables x and y in the formula $xy = 12$ for a range of values of x from −6 to +6.
(*b*) From your graph read the values of y when (*i*) $x = 1{\cdot}5$, (ii) $x = 2{\cdot}4$, (iii) $x = -3{\cdot}5$.

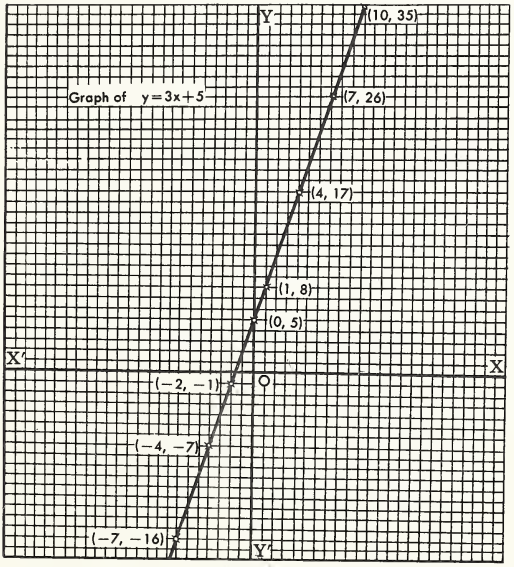
Graphs of Algebraic Equations

The equation $y = 3x + 5$ expresses a relationship between the variables x and y. By assigning any value to x and substituting this value in the equation,

a corresponding value of y can be determined. Arrange the sets of values in a table such as the following:

x	10	7	4	1	0	−2	−4	−7
y	35	26	17	8	5	−1	−7	−16

Choose a rectangular set of axes with the x-axis horizontal. Using the sets of values from the table as co-ordinates, plot the points. Join the plotted points to complete the graph for the desired range.



Check: Select any other point on the graph. Substitute its co-ordinates for x and y in the equation. They should satisfy the equation. Test other points in the same manner.

The Fundamental Relationship between Equations and Their Graphs

When an equation is represented graphically, all sets of simultaneous values of the variables which satisfy the equation are co-ordinates of points on the graph; and, conversely, the co-ordinates of any point on the graph when substituted for the variables in the equation satisfy the equation.

Exercise B

Draw the graph for each of the following formulas:

1. $P = 4s$ for a range of s from -10 to $+10$.

2. $C = \pi d$ for a range of d from 0 to 10. Read from the graph the value of C which corresponds to each of the following values of d:
   (*a*) 0·8, (*b*) 3·2, (*c*) 7, (*d*) 8·8.

3. $C = 2{\cdot}54i$ for a range of i from 0 to 12. Read from the graph the value of C which corresponds to the following values of i:
   (*a*) 2·5, (*b*) 5, (*c*) 8·75, (*d*) 11·5.

4. $A = 0{\cdot}7854d^2$ for a range of d from 0 to 10.

5. $s = \sqrt{a}$ for a range of a from 0 to 100. Read from the graph the value of s when a is (i) 3, (ii) 5, (iii) 40, (iv) 89.

6. Draw the graphs of the following equations:
   (*a*) $y = x - 5$. (*b*) $y = 5 - 2x$.
   (*c*) $y = \dfrac{6 + 3y}{4}$. (*d*) $x + 4y = 8$.
   (*e*) $3x = 4y$. (*f*) $2x - 3y = 12$.

7. Draw the graphs of the following equations:
   (*a*) $x + y = 8$. (*b*) $x - y = 4$.
   (*c*) $x + 3y = 12$. (*d*) $x - 5y = 25$.
   (*e*) $4x - 3y = 24$. (*f*) $\frac{1}{2}x = y$.

8. Draw and describe in words the graphs of each of the following equations:
   (*a*) $x = 5$. (*b*) $y = 4$. (*c*) $x + 3 = 0$.
   (*d*) $y + 7 = 0$. (*e*) $x = y$. (*f*) $y = -x$.

Simultaneous Equations Solved Graphically

Example: Find a set of values of x and y which will satisfy both of the equations $2x - y = 11$ and $x - 2y = 10$.

Solution: Solve the equation $2x - y = 11$ for y and obtain $y = 2x - 11$. Make a table of sets of corresponding values of x and y.

x	-3	-2	0	4	8	10
y	-17	-15	-11	-3	5	9

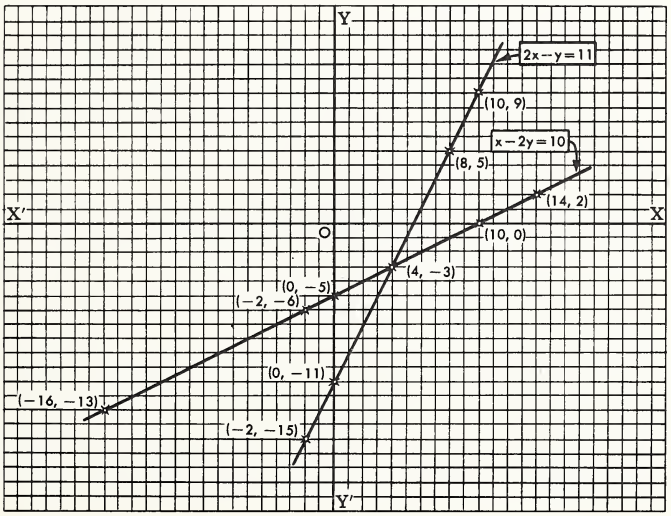
Draw the graph of $2x - y = 11$.

Solve the equation $x - 2y = 10$ for y and obtain $y = \dfrac{x - 10}{2}$.

Make a table of sets of corresponding values of x and y.

x	-3	-2	0	4	10	14
y	$-6\frac{1}{2}$	-6	-5	-3	0	2

Draw the graph of $x - 2y = 10$.



The straight lines intersect at the point (4, −3). These values for x and y respectively satisfy both equations. The equations have been solved graphically to obtain $x = 4$, $y = -3$.

Exercise B

Draw the graph of each of the following sets of equations and from your graph find and record a set of values of x and y which will satisfy both equations simultaneously.

1. $x + y = 5$.
$x - y = 3$.

2. $2x + y = 7$.
$3x + 2y = 12$.

3. $x + 2y = 3.$
$2x + y = 9.$

4. $2x - 4y = 8.$
$x - y = 1.$

5. $2x + 3y = 5.$
$x + 2y = 4.$

6. $2x - 3y = 1.$
$3x + y = 7.$

7. $4x + 3y = 10.$
$2y - x = 3.$

8. $4x - 3y = 1.$
$2x + y = 8.$

9. $3x + 4y = 13.$
$6x - 4y = -1.$

10. $y = 4x.$
$x - 3y = 0.$

Simultaneous Equations Solved Algebraically

When a problem contains two unknown measures, the solution usually involves simultaneous equations. The following problem will illustrate:

Two customers of the same fruit store compared notes to determine the prices charged for fruit. The first customer paid 34 cents for 3 pears and 4 oranges. The second paid 41 cents for 6 pears and 3 oranges. What was the unit price for the pears and for the oranges?

Let the price of each pear in cents be x.

Let the price of each orange in cents be y.

The amount paid by the first customer in cents is $3x + 4y$.

Therefore $3x + 4y = 34$.....(1)

The amount paid by the second customer in cents is $6x + 2y$.

Therefore $6x + 2y = 41$.....(2)

To solve a pair of simultaneous equations, algebraically, it is necessary to obtain from them one equation in one unknown. In this case this can be done as follows:

Multiply equation (1) by 2, in order to obtain the same coefficients of x in both equations.

$$6x + 8y = 68$$
$$6x + 2y = 41$$

Subtract to eliminate x $\quad 6y = 27$

Solve for y $\quad y = 4\frac{1}{2}.$

To obtain the corresponding value of x, substitute $y = 4\frac{1}{2}$ in (1).

Then $\quad 3x + 4 \times 4\frac{1}{2} = 34.$

Solve for x $\quad x = 5\frac{1}{3}.$

The solution of the equations is $x = 5\frac{1}{3}$, $y = 4\frac{1}{2}$.

The unit price of the pears in cents is $5\frac{1}{3}$.

The unit price of the oranges in cents is $4\frac{1}{2}$.

As there are several steps in the solution of simultaneous equations, a systematic arrangement is essential. A suitable arrangement is shown in the following example:

Example: Solve and verify $3x + 5y = 21.$
$7x - 3y = 5.$

Solution:

$3x + 5y = 21 \ldots .(1)$
$7x - 3y = 5 \ldots .(2)$

Mult. (1) by 3. $9x + 15y = 63.$
Mult. (2) by 5. $35x - 15y = 25.$
Add $44x = 88.$
$x = 2.$

Sub. $x = 2$ in (1) $6 + 5y = 21.$
$5y = 15.$
$y = 3.$

$x = 2, y = 3.$

Verification:

Put $x = 2$ and $y = 3$ in (1).

L.S. $= 3x + 5y$
$= 3 \times 2 + 5 \times 3$
$= 6 + 15$
$= 21.$
R.S. $= 21.$

Put $x = 2$ and $y = 3$ in (2).

L.S. $= 7x - 3y$
$= 7 \times 2 - 3 \times 3$
$= 14 - 9$
$= 5.$
R.S. $= 5.$

Therefore $x = 2$, $y = 3$ satisfy both equations.

It will be noted that after suitable preparation of the equations one of the unknowns may be eliminated by the application of the axiom of subtraction (or addition).

This method is usually designated as 'Method of elimination by addition or subtraction'.

The following examples will illustrate a second method of elimination of one of the unknowns.

Example 1: A football team sold 350 student and adult tickets to their football game. The student tickets cost 50 cents each and the adult tickets 75 cents each. How many tickets of each kind did the team sell if they received a total of $200?

Solution: Let x represent the number of student tickets sold.
Let y represent the number of adult tickets sold.
Then $.5x$ dollars will represent the value of the student tickets.
$.75y$ dollars will represent the value of the adult tickets.
$(x + y)$ will represent the number of tickets sold.
$(.5x + .75y)$ dollars will represent the number of dollars received.

Then $x + y = 350 \ldots .(1)$
$.5x + .75y = 200 \ldots .(2)$

Solve (1) for x and obtain $\ldots .x = 350 - y \ldots .(3)$

Substitute $350 - y$ for x in (2) and obtain $.5(350 - y) + .75y = 200$
$175 - .5y + .75y = 200$
$.25y = 25$
$y = 100$

To obtain the corresponding value of x substitute $y = 100$ in (3).
Then $x = 350 - 100$
$x = 250$
There were 250 student tickets sold.
There were 100 adult tickets sold.

Example 2:

Solve by substitution and verify $2x - y = 8$.
$3x + 2y = 5$.

Solution:

$2x - y = 8 \ldots .(1)$
$3x + 2y = 5 \ldots .(2)$

Solve for y in (1) $y = 2x - 8 - (3)$
Substitute in (2) $3x + 2(2x - 8) = 5$
$3x + 4x - 16 = 5$
$7x = 21$
$x = 3$
Substitute $x = 3$ in (1) $2 \times 3 - y = 8$
Solve for y $6 - y = 8$
$y = -2$

$x = 3 \quad y = -2$

Verification:

Put $x = 3$ and $y = -2$ in (1)
L.S. $= 2 \times 3 - (-2)$
$= 6 + 2$
$= 8$
R.S. $= 8$

Put $x = 3$ and $y = -2$ in (2)
L.S. $= 3 \times 3 + 2(-2) = 5$
$= 9 - 4$
$= 5$
R.S. $= 5$

$x = 3, y = -2$ satisfies both equations.

This method for elimination of one of the unknowns is usually designated as elimination by substitution.

Exercise B

Solve and verify:

1. $x + y = 5$.
$x - y = 4$.

2. $3x + 4y = 8$.
$x + 4y = 6$.

3. $5x - 3y = 1$.
$x + 3y = 5$.

4. $3x - 4y = 0$.
$6x + 4y = 3$.

5. $3x + 4y = 10$.
$3x + 2y = 8$.

6. $5x - 6y = -7$.
$3x + 6y = 15$.

Solve, using the method of substitution to eliminate one of the unknowns:

7. $5x - 3y = 1$.
$x + 4y = 14$.

8. $x = 5y + 2$.
$2x - y = -5$.

9. $y = 3x - 4$.
$2x + y = 11$.

10. $5a + 4b = 2$.
$4a + 5b = 7$.

11. $y = 2x + 5$.
$4x + 3y = 20$.

12. $3x + 2y = 12$.
$5x + 4y = 22$.

Solve, using the method considered to be the more suitable.

13. $2a + 3b = -6$.
$3a - 4b = 8$.

14. $5x + y = 1$.
$2x - y = -8$.

15. $6m - 7n = 1$.
$7m - 8n = 2$.

16. $x = 3y - 7.$
$x = -y + 5.$

17. $y = 5x + 4.$
$y = 4x + 6.$

18. $3y = 4x + 2.$
$3y = 2x - 6.$

19. Find a pair of numbers which have a sum of 35 and a difference of 1.

20. Find a pair of numbers which have a sum of -3 and a difference of $+13$.

21. If the sides of an equilateral triangle are $2x + 7$, $4y + x$ and $5y + 2$, find the length of each side.

22. A Hi-fi unit and 50 records cost \$299.00. The same unit and 80 records cost \$328.40. Find the cost of the unit and of one record.

23. If 5 pears and 6 apples cost 38 cents and 6 pears and 12 apples cost 60 cents, find the cost of each article.

24. John has \$1.70 more than Hugh. If John gives Hugh 10 cents, he will have twice as much as Hugh. Find how much each has.

25. Find a fraction such that if the terms are each increased by 1 the fraction becomes $\frac{7}{8}$, but if the terms are each reduced by 1 the fraction is equal to $\frac{6}{7}$.

26. Twenty coins, all dimes and nickels, have a value of \$1.65. Find the number of each.

27. Solve for a and b: $a + \frac{1}{2}b = 13.$

$$\frac{a}{3} - b = 2.$$

28. Solve $\frac{3}{4}x - \frac{1}{3}y = 9.$
$-x + 2y = 2.$

29. Solve $\frac{x}{2} - \frac{y}{3} = 3x - 2y - 5.$

$$\frac{2x}{5} - \tfrac{1}{4}y = 3x - 2y - 5.$$

30. Solve $0{\cdot}24x - 0{\cdot}4 = 0{\cdot}8y.$
$0{\cdot}01x + 0{\cdot}1y = 0{\cdot}03.$

31. In the formula $A = pq + r$, find p and r if $A = 31$ when $q = 9$ and $A = 49$ when $q = 15$.

32. If $ax + by = 10$ either when $x = 1$ and $y = 2$ or when $x = 4$ and $y = 10$, find the values of a and b.

33. In the formula $v = u + at$, it is known that $v = 74$ when $t = 2$ and $v = 170$ when $t = 5$. Find the value of a.

34. At a special sale, a grocer gave 3 lb. coffee and 5 lb. sugar for \$3.00, or 2 lb. coffee and 3 lb. sugar for \$1.96. What was the price per pound of each?

35. A dry-cleaning firm charged \$1.05 for laundering 3 shirts and 6 handkerchiefs. For the next bundle containing 5 shirts and 3 handkerchiefs, the cost was \$1.40. What was charged for each article?

C

36. A plane leaves its aerodrome and travels due east at 240 m.p.h. It is pursued by a second plane which passed the aerodrome $\frac{1}{2}$ hour later travelling at 360 m.p.h. At what time after its departure and how far from the aerodrome will the first plane be when overtaken?

37. (*a*) A man starts out at 9 a.m. and walks steadily at 3 miles per hour. Draw a graph showing his distance from the starting-point during the next 5 hours.

(*b*) Using the same reference axes and units, draw a graph showing the distance from the starting-point of a cyclist who left two hours later than this man and followed him at a rate of 6 miles per hour.

(*c*) At what time does the cyclist overtake the walker?

(*d*) What distance lies between them at (i) noon, (ii) 1 p.m.?

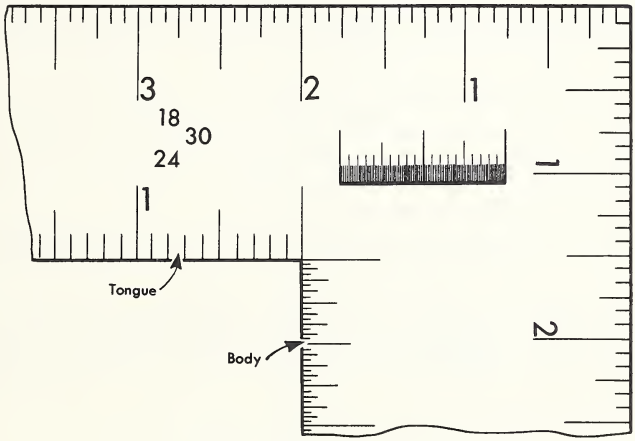
38. By a graphical method find the time and the distance from Toronto at which two cars will meet if car **A** leaves Toronto at 9 a.m. for Chatham at 40 m.p.h. and car **B** leaves Chatham at 10 a.m. for Toronto at 50 m.p.h. Assume that the distance from Toronto to Chatham is 200 miles.

CHAPTER 17

Scales and Scale Reading

Between the time a product starts as a piece of raw material and the time it emerges as a finished article, measuring tools of various kinds are essential to its manufacture. Measuring instruments such as rulers, micrometers, calipers, protractors and gauges are equipped with scales to make accurate measurement possible.

Most scales found on measuring instruments are designed according to one of four systems. Three of these can be found on the back of a steel square, a section of which is illustrated below.



Steel Square

(1) In the **decimal system**, the basic unit is divided into tenths, hundredths, etc. This system is illustrated by the scale on the inside edge of the tongue of the steel square.

(2) In the **binary system**, the basic unit is divided into halves, quarters, eighths, etc. Such a scale is on the inside edge of the body of the steel square.

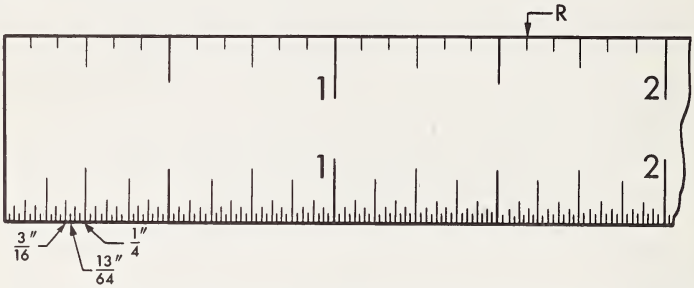
(3) In the **duodecimal system**, the basic unit is divided into twelfths. The scale on the outside edge of the tongue and the body is a duodecimal scale.

(4) In a combination of the decimal and binary systems, the basic unit is divided into tenths and each tenth into halves and quarters. The main scales on the vernier caliper and on the micrometer are of this type.

Scale Reading

An example will illustrate a procedure by which a reading may be taken from an unfamiliar scale.

Example: Find the reading marked R on the enlarged section of the duodecimal scale shown below.



Solution: The basic unit is 1 inch. Each inch is divided into halves and quarters.

The smallest division is $\frac{1}{3} \times \frac{1}{4}$ inch or $\frac{1}{12}$ inch.

The reading in inches is $1 + \frac{1}{2} + \frac{1}{12} = 1\frac{7}{12}$.

A reading which involves the smaller scale divisions may be located on the scale more easily if it is first expressed in terms of a larger adjacent division.

For example, it will help to locate the position of $\frac{13}{64}$ on the binary scale if it is expressed either as $\frac{3}{16} + \frac{1}{64}$ or as $\frac{1}{4} - \frac{3}{64}$. Locate the larger division first and then move right, or left, the number of smaller divisions that are indicated.

Exercise

A

1. (*a*) Express each of the following scale readings in quarters increased or decreased by $\frac{1}{16}$: $\frac{11}{16}$, $\frac{5}{16}$, $\frac{13}{16}$, $\frac{3}{16}$.

(*b*) Express each of the following scale readings in eighths increased or decreased by $\frac{1}{32}$: $\frac{21}{32}$, $\frac{29}{32}$, $\frac{19}{32}$, $\frac{13}{32}$, $\frac{27}{32}$, $\frac{11}{32}$.

2. Express each of the following scale readings in sixteenths increased or decreased by sixty-fourths: $\frac{43}{64}$, $\frac{21}{64}$, $\frac{37}{64}$, $\frac{35}{64}$, $\frac{41}{64}$.

3. Express each of the following scale readings in eighths increased or decreased by thirty-seconds: $\frac{11}{32}$, $\frac{25}{32}$, $\frac{19}{32}$, $\frac{7}{32}$.

B

4. Using 4 inches to represent 1 inch, make a sketch of a binary scale divided into 64ths of an inch. Mark and label each of the following readings:

(*a*) $\frac{3}{16}$, $\frac{7}{16}$, $\frac{11}{16}$; (*b*) $\frac{3}{32}$, $\frac{9}{32}$, $\frac{19}{32}$, $\frac{27}{32}$; (*c*) $\frac{3}{64}$, $\frac{9}{64}$, $\frac{21}{64}$, $\frac{31}{64}$, $\frac{41}{64}$.

5. Using a line 5 inches long to represent 1 inch, make a sketch of a scale divided into tenths and hundredths. On this scale mark and label the following readings:

(*a*) 0·30, 0·35, 0·45, 0·70, 0·95; (*b*) 0·02, 0·08, 0·22, 0·27, 0·81;
(*c*) the approximate positions of 0·125, 0·537, 0·775.

6. Using a line 5 inches long to represent 1 inch, make a sketch of a scale divided into tenths and fortieths. On this scale mark and label the following readings:

(*a*) 0·2, 0·55, 0·75, 0·80;
(*b*) 0·05, 0·125, 0·275, 0·325, 0·475, 0·825, 0·975;
(*c*) the approximate positions of 0·04, 0·17, 0·51, 0·73, 0·112, 0·362, 0·787, 0·937.

7. Using a line 3 inches long to represent 1 inch, make a sketch of a scale divided into quarters and twelfths. On this scale mark and label the following readings:

(*a*) $\frac{5}{12}$, $\frac{7}{12}$, $\frac{11}{12}$;
(*b*) the approximate positions of $\frac{1}{24}$, $\frac{5}{24}$, $\frac{7}{24}$, $\frac{5}{8}$, $\frac{13}{16}$, 0·7, 0·9.

Approximate Scale Readings

A calculated measurement may contain a fraction different from that found on the scale. It is then necessary to find the scale division nearest to this measure.

Example 1: On the scale divided into 64ths, what scale division is nearest to $\frac{3}{7}$?

Solution:
$$\frac{3}{7} = \frac{\frac{64}{7} \times 3}{64}$$
$$= \frac{27\frac{3}{7}}{64}.$$

The new denominator must be 64. To obtain the new denominator, 7 is multiplied by $\frac{64}{7}$. To keep the value of the fraction the same the numerator, 3, must be multiplied by $\frac{64}{7}$.

Therefore, the scale division nearest to $\frac{3}{7}$ is $\frac{27}{64}$.

Example 2: A piece of stock is to be reduced in diameter from $\frac{1}{2}''$ to 0·413″. A pair of calipers used for checking purposes is set at the next 64th greater than 0·413. At what scale reading should the calipers be set?

Solution 1: The scale reading in 64ths is $\frac{0{\cdot}413 \times 64}{64}$

$$= \frac{26{\cdot}432}{64}.$$

The caliper setting is $\frac{27}{64}$.

Solution 2: In the table of decimal equivalents locate the fraction which is just greater than 0·413. In this case it is 0·421875 or $\frac{27}{64}$. Therefore, the caliper setting is $\frac{27}{64}$.

Exercise

A

1. Change

(*a*) $\frac{3}{4}$ to 16ths, (*b*) $\frac{5}{8}$ to 32nds,

(*c*) $\frac{15}{16}$ to 64ths, (*d*) $\frac{7}{8}$ to 64ths.

2. (*a*) Change $\frac{1}{7}$ to the nearest 16th.

(*b*) Change $\frac{1}{3}$ to the nearest 16th.

(*c*) Change $\frac{1}{9}$ to the nearest 32nd.

3. Using a table of decimal equivalents, find the nearest reading on a 64th scale for

(*a*) 0·138, (*b*) 0·258, (*c*) 0·663, (*d*) 0·789,

(*e*) $\frac{4}{5}$, (*f*) $\frac{1}{7}$, (*g*) $\frac{13}{25}$, (*h*) $\frac{5}{12}$.

B

4. Find the division on a 64th scale which is nearest to

(*a*) $\frac{1}{3}$, (*b*) $\frac{5}{7}$, (*c*) $\frac{5}{6}$, (*d*) $\frac{7}{20}$, (*e*) $\frac{51}{70}$.

5. Find the division on a 32nd scale which is nearest to

(*a*) $2\frac{1}{5}$, (*b*) $3\frac{2}{7}$, (*c*) $5\frac{7}{100}$, (*d*) $251\frac{9}{40}$.

6. Find the division on a 64th scale which is nearest to

(*a*) 0·343, (*b*) 0·57, (*c*) 0·056, (*d*) 3·88.

7. Find the division on the 64th scale which is just greater than

(*a*) $\frac{6}{7}$, (*b*) $3\frac{5}{9}$, (*c*) 0·667, (*d*) 0·910.

8. Calculate the error when the nearest 64th scale division is used in place of each of the following:

(*a*) 0·146, (*b*) 0·850, (*c*) 0·628, (*d*) 0·800.

9. Find, to the nearest 64th of an inch, the length of one side of a square the area of which is

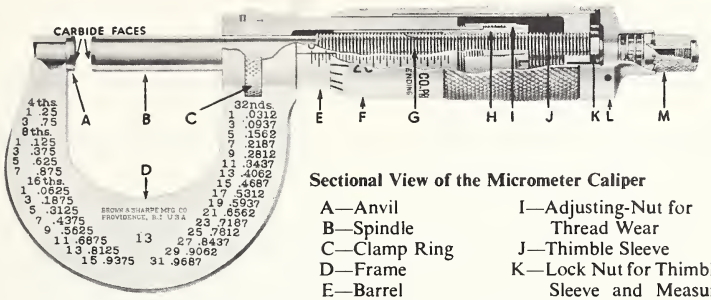
(*a*) 75 sq. in., (*b*) 125 sq. in., (*c*) 10 sq. in.

10. Express, to the nearest eighth of an inch, the length of a rafter which was found by the calculation to be

(*a*) 13·612 feet, (*b*) 8·562 feet, (*c*) 9·173 feet.

The Micrometer Caliper

One revolution of the thimble of a micrometer opens the jaws a distance of one scale division because the micrometer has a threaded screw with a



Courtesy Brown & Sharpe Mfg. Co., Providence, Rhode Island

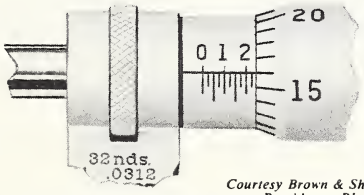
Sectional View of the Micrometer Caliper

A—Anvil	I—Adjusting-Nut for Thread Wear
B—Spindle	J—Thimble Sleeve
C—Clamp Ring	K—Lock Nut for Thimble Sleeve and Measuring-Screw
D—Frame	L—Thimble Cap
E—Barrel	M—Ratchet Stop
F—Thimble	
G—Measuring-Screw	
H—Fixed Nut	

Micrometer Caliper

pitch equal to one of the scale divisions on the barrel. The scale on the thimble indicates fractional parts of a turn.

To Read a Micrometer



Courtesy Brown & Sharpe Mfg. Co., Providence, Rhode Island

Micrometer Caliper Scale

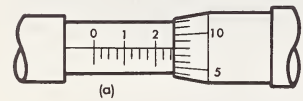
One inch on the barrel is divided into tenths and fortieths. Each numbered division on the barrel indicates tenths of an inch. Each small division indicates $\frac{1}{40}$″ or 0·025″. The bevelled edge of the thimble is divided into 25 parts. Each small division on the thimble indicates $\frac{1}{25}$ of $\frac{1}{40}$ or 0·001″. The reading illustrated above may be stated in this manner:

2 the last number visible on the barrel gives	·200
1 additional small division gives	·025
16 divisions on the thimble gives	·016
Therefore, the measurement in inches is	·241.

Exercise

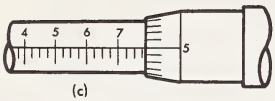
A

1. State the readings for the following micrometer settings:

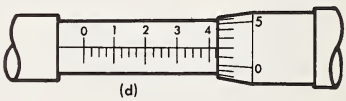


(a)

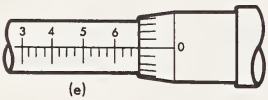
(b)



(c)

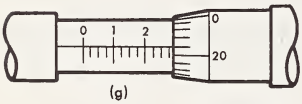


(d)

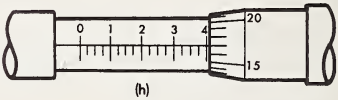


(e)

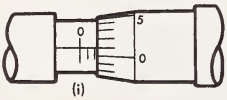
(f)



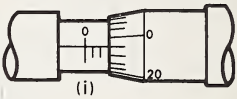
(g)



(h)



(i)



(j)

B

2. Construct sketches to illustrate the following readings or settings on a micrometer:

(*a*) 0·347, (*b*) 0·545, (*c*) 0·762, (*d*) 0·057,
(*e*) 0·006, (*f*) 0·489, (*g*) 0·455, (*h*) 0·716,
(*i*) 0·424, (*j*) 0·112, (*k*) 0·625, (*l*) 0·023.

3. Draw a diagram to show the setting on a micrometer for the reading nearest to

(*a*) $\frac{3}{16}$, (*b*) $\frac{2}{3}$, (*c*) $\frac{5}{8}$, (*d*) $\frac{15}{16}$.

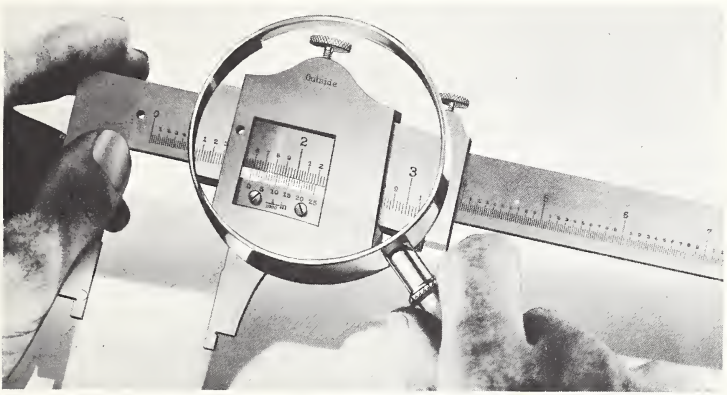
4. A piece of work in a lathe has to be turned to 0·621″ in diameter. Show, by diagram, the setting that should be made on the micrometer so that the final diameter may be checked.

5. The thickness of No. 3 gauge steel is 0·2391″. Show the setting on the micrometer to check this thickness.

6. Sketch the setting on a thousandths micrometer for checking the thickness of sheet steel according to the Brown and Sharpe gauge sizes as follows:

Gauge	*Thickness in inches*	*Gauge*	*Thickness in inches*
0	0·3249	20	0·0320
5	0·1819	25	0·0179
10	0·1019	30	0·0100
15	0·0571	35	0·0056

The Vernier



Courtesy Brown & Sharpe Mfg. Co., Providence, Rhode Island

Vernier Caliper

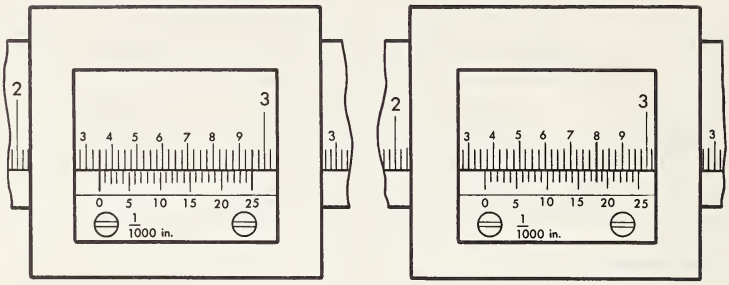
The instrument shown in the diagram is called a vernier caliper. It has a fixed scale divided into tenths and fortieths of an inch. The vernier device is a sliding scale attached to the fixed scale with the zero mark on the vernier used as the indicator.

The setting in diagram 1 on the next page shows the indicator directly opposite a scale division. The reading 2·350 can be taken directly.

From the diagram it can be seen that 25 divisions on the vernier scale are equivalent to 24 divisions on the fixed scale.

The length of 24 divisions in inches is $24 \times 0{\cdot}025 = 0{\cdot}600$.

The length of 1 vernier division in inches is $\frac{1}{25} \times 0{\cdot}600 = 0{\cdot}024$. Therefore, the difference in length between a division on the scale and on the vernier is 0·001 inch.



Vernier Setting 1 **Vernier Setting 2**

In diagram 2, the indicator has just passed 2·350. Since the eighteenth vernier line coincides with a scale division, then 18 × 0·001 or 0·018 must be added to obtain the scale reading. The reading is then 2·350 + 0·018 = 2·368.

Example: What is the reading on a fortieths scale equipped with a vernier attachment when the number of main divisions passed is 5, additional tenths 4, additional fortieths 3 and the coinciding vernier line is 8?

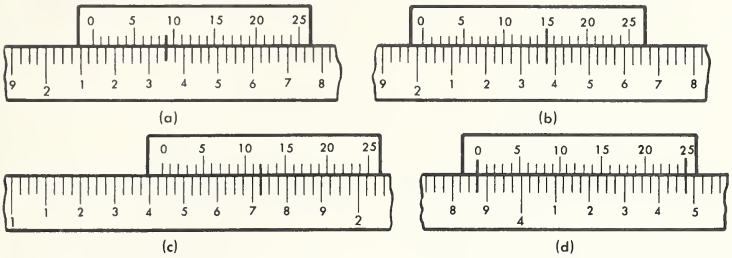
Solution: The reading is 5· + 0·4 + 3 × 0·025 + 8 × 0·001 = 5·483.

Exercise B

1. Find the readings indicated by each of the following settings on the vernier caliper:

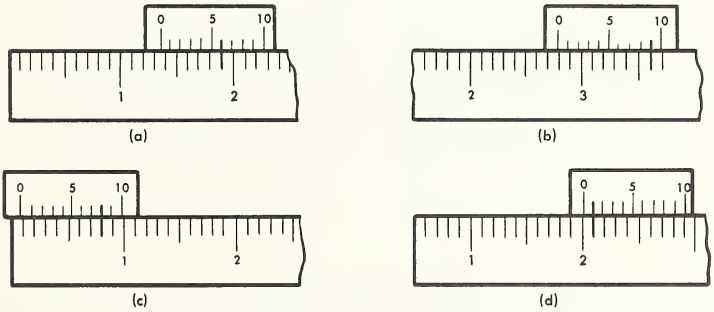
	Number of Main Divisions	*Additional Tenths*	*Additional Fortieths*	*Coinciding Vernier Line*
(*a*)	3	4	2	12
(*b*)	2	1	1	3
(*c*)	1	6	3	4
(*d*)	0	3	2	4
(*e*)	4	5	0	21
(*f*)	2	0	3	17
(*g*)	5	0	0	2
(*h*)	6	7	1	18

2. State the readings indicated by the sketch of each of the following vernier caliper settings:



3. Draw a simple sketch to indicate the setting on a vernier caliper for each of the following readings:
(*a*) 0·387, (*b*) 3·416, (*c*) 2·239, (*d*) 4·563.

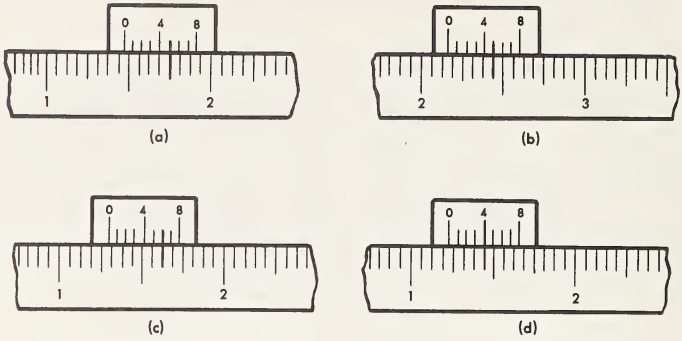
4. The sketches below illustrate a vernier used with a scale divided into tenths. Ten divisions on the vernier are equivalent to nine divisions on the fixed scale. Explain how this vernier can be used to obtain a reading to the nearest hundredth. What is the reading on each of the settings?



5. Draw a simple sketch of a decimal scale and vernier to indicate the setting for each of the following readings:
(*a*) 4·43, (*b*) 2·67, (*c*) 1·32, (*d*) 0·45.

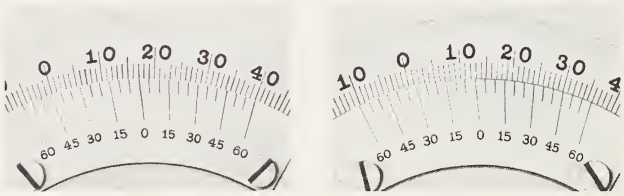
C

6. On the next page is illustrated a vernier used with a sixteenth scale. Eight divisions on the vernier are equivalent to seven divisions on the fixed scale. Explain how this vernier can be used to obtain the reading to the nearest $\frac{1}{128}$". What is the reading indicated by each setting?



7. Draw a simple sketch of a $\frac{1}{16}''$ scale and vernier to indicate the setting for each of the following readings:
   (*a*) $1\frac{31}{128}$, (*b*) $1\frac{5}{64}$, (*c*) $1\frac{27}{32}$, (*d*) $1\frac{39}{64}$.

The Vernier on a Protractor Scale



Courtesy Brown & Sharpe Mfg. Co., Providence, Rhode Island

Vernier Protractor Setting 1 **Vernier Protractor Setting 2**

The main division on the protractor scale is one degree. There are 12 divisions on the vernier but it will help us to understand it better if we consider each of these divisions divided into two parts. From setting 1 it can be seen that 24 parts on the vernier correspond to 23 parts on the scale.

The angle in degrees indicated by 1 vernier part is $\frac{1}{24}$ of 23. The difference in angle indicated by each division of the scale and a part on the vernier is $\frac{1}{24}$ of 1 degree or $2\frac{1}{2}$ minutes. The difference indicated by each division (2 parts) on the vernier is 5 minutes.

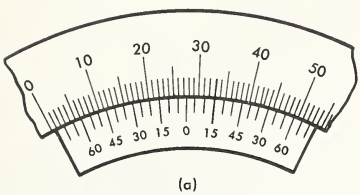
In diagram 2, the indicator has passed 12 degrees. The 50-minute line on the vernier coincides with a line on the scale. Hence the reading is 12° 50′.

A double vernier scale is needed so that angles may be measured in either direction from zero.

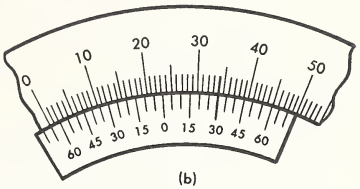
Exercise

A

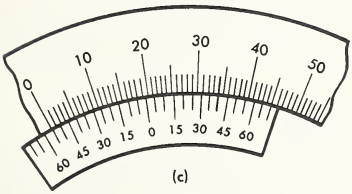
1. Give the protractor readings for each of the sketches shown below:



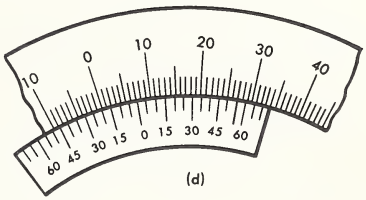
(a)



(b)



(c)



(d)

B

2. With the aid of compasses, draw a simple sketch of a protractor scale with vernier to indicate the setting for each of the following readings:

(*a*) 45° 15′, (*b*) 30° 20′, (*c*) 15° 55′,
(*d*) 25° 0′, (*e*) 114° 35′, (*f*) 0° 45′.

CHAPTER 18

Shop Calculations

Cutting Speed

In modern industry, most of the grinding, cutting and drilling that is necessary to shape materials into useful products is done by power tools.

The speed at which the cutting edge of the tool passes through the material is called the **cutting speed** and is expressed in feet per minute.

In practice, the actual cutting speed depends upon such factors as the type of the job, the condition of the material, the kind and type of cutting tool and the finish required. To obtain the cutting speed for a specific job, the operator may consult a handbook.

Cutting Speed of Grinders

In one turn, the grinding wheel cuts a distance of one circumference. In one minute, the distance in contact would be the product of the circumference and the r.p.m.

Example 1: Calculate the cutting speed of a 7-inch grinder which rotates at a speed of 2,000 r.p.m.

Solution: The circumference of the grinder in feet is $\frac{3{\cdot}14 \times 7}{12}$.

The calculated cutting speed in f.p.m. is $\frac{3{\cdot}14 \times 7 \times 2{,}000}{12}$

$= 3{,}663.$

Example 2: Calculate the r.p.m. at which a 10″ emery wheel should be turned so that the surface speed is 5,000 f.p.m.

Algebraic Solution: Let the r.p.m. of the grinder be x.

Then, the cutting speed in f.p.m. is $\frac{3{\cdot}14 \times 10}{12} \times x = 2{\cdot}617x.$

But, the cutting speed in f.p.m. is 5,000.

Therefore, $2{\cdot}617x = 5{,}000.$

Solving we get $x = 1{,}911.$

Therefore, the calculated r.p.m. of the grinder is 1,911.

Exercise B

1. Calculate the cutting speed of an 8″ grinder which rotates at 2,400 r.p.m.
2. Calculate the rim speed of a 20″ wheel which is rotating at 900 r.p.m.
3. A 6″ pulley is attached to a motor with a speed of 1,400 r.p.m. Calculate the speed of a point on the belt in f.p.m. if there is no slipping.
4. Calculate, to two significant digits, the cutting speeds for each of the following grinders:

   (*a*) diameter 4″, r.p.m. 5,000; (*b*) diameter 22″, r.p.m. 800;
   (*c*) diameter 16″, r.p.m. 1,300; (*d*) diameter 12″, r.p.m. 2,700.
5. Calculate the r.p.m. at which an 8″ grinding wheel should be rotated to give a cutting speed of

   (*a*) 4,000 f.p.m., (*b*) 6,000 f.p.m.
6. Calculate the r.p.m. at which a 6″ pulley should be rotated to give a belt speed of 400 f.p.m.
7. Calculate the maximum r.p.m. of a flywheel which has a diameter of 2′ 6″ if its rim speed must not exceed one mile a minute.
8. Calculate, to two significant digits, the r.p.m. for each of the grinders as follows:

   (*a*) diameter 4″, cutting speed 6,000 f.p.m.;
   (*b*) diameter 22″, cutting speed 4,500 f.p.m.;
   (*c*) diameter 30″, cutting speed 5,500 f.p.m.;
   (*d*) diameter 2″, cutting speed 5,000 f.p.m.
9. Calculate the diameter of a grinding wheel which should be used on a shaft turning at 3,000 r.p.m. if the desired cutting speed is 5,500 f.p.m.
10. If a grinding speed of 4,000 f.p.m. is recommended and the r.p.m. of the shaft is 850, find the diameter of the wheel which will give this speed.
11. If the speed of a conveyor belt is to be 100 f.p.m. and the speed of the driving pulley attached to it is 36 r.p.m., calculate the diameter of the pulley.
12. Grinder wheels are made with diameters in inch sizes from 1″ to 8″ inclusive and in even-numbered inches from 8″ to 36″. Select a standard size grinder wheel to use in each of the following situations:

   (*a*) cutting speed 5,000 f.p.m., r.p.m. 3,183;
   (*b*) cutting speed 4,000 f.p.m., r.p.m. 700;
   (*c*) cutting speed 5,500 f.p.m., r.p.m. 7,000;
   (*d*) cutting speed 6,000 f.p.m., r.p.m. 700.
13. Find the diameter, to the nearest inch, of a pulley turning at 300 r.p.m. which will have a rim velocity of 1,500 f.p.m.

14. What should be the size, to the nearest inch, of a pulley to turn with a speed of 2,400 r.p.m. if it is driven by a belt which has a velocity of 4,000 f.p.m.?

Cutting Speed of a Drill Press and Milling Machine

For work in the drill press, the cutting speed is the rate at which the extreme edges of the drill cut the work.

For work in the milling machine, the cutting speed is the rate at which the outer corners of the cutting tool cut the work.

Example 1: What is the cutting speed of a $\frac{5}{8}''$ drill turning at 480 r.p.m.?

Solution: The circumference in feet is $\frac{3{\cdot}14 \times \frac{5}{8}}{12}$.

The calculated cutting speed in f.p.m. is $\frac{3{\cdot}14 \times \frac{5}{8} \times 480}{12}$

$= 78{\cdot}5.$

Example 2: At what r.p.m. should a 4″ cutter be operated in the milling machine if the cutting speed is not to exceed 200 f.p.m.?

Algebraic Solution: Let the r.p.m. of the cutter be x.

Then, the cutting speed in f.p.m. is $\frac{3{\cdot}14 \times 4 \times x}{12} = 1{\cdot}05x.$

But, the cutting speed in f.p.m. is 200.

Therefore, $1{\cdot}05x = 200.$

Solving we get, $x = 190{\cdot}4.$

Therefore, the r.p.m. should not exceed 190.

The Feed of a Drill

The feed of a drill is the distance in inches that the drill advances in one revolution. It is usually expressed in thousandths of an inch. The feed depends upon the size of the drill and the type of the material. It is necessary to know the feed in order to determine the time necessary to do a drilling operation.

Example: Find the time required to drill separately 4 holes, each $\frac{1}{4}''$ in diameter and $2\frac{1}{2}''$ deep if the feed of the drill is 0·012″ and the r.p.m. is 720. Allow 60 seconds for adjusting the work.

Solution: The number of revolutions required for 1 hole is $\frac{2{\cdot}5}{0{\cdot}012}$.

The time in minutes for one hole is $\frac{2{\cdot}5}{0{\cdot}012 \times 720}$.

The time in minutes for 4 holes is $\frac{4 \times 2{\cdot}5}{0{\cdot}012 \times 720}$

$= 1{\cdot}16.$

The time for drilling and adjusting is 2·16 minutes or 2 minutes 10 seconds.

Exercise

B

1. Find the cutting speed of a $\frac{1}{8}''$ drill rotating at 1,750 r.p.m.
2. Find the r.p.m. at which a $\frac{3}{4}''$ drill will have a cutting speed of 140 f.p.m.
3. A 6″ milling cutter is rotating at 108 r.p.m. Find the cutting speed.
4. Find the r.p.m. to set a milling machine with a 6″ cutter if a cutting speed of 250 f.p.m. is required.
5. One-inch holes are being drilled in a cast-iron plate at a cutting speed of 60 f.p.m. What should be the r.p.m. of the drill?
6. Calculate the time required to drill each of the following holes:
   (*a*) 4″ deep with a 1″ drill, feed of 0·015″ at 840 r.p.m.;
   (*b*) $2\frac{3}{4}''$ deep with a $\frac{1}{2}''$ drill, feed of 0·006″ at 1,000 r.p.m.;
   (*c*) $3\frac{1}{2}''$ deep with a $\frac{3}{4}''$ drill, feed of 0·010″ and a cutting speed of 60 f.p.m.
7. According to the information in a machinist handbook, a number **A** drill should rotate at 654 r.p.m. to give a cutting speed of 40 f.p.m. Assuming that this data is correct, calculate the diameter of a number **A** drill.

The Cutting Speed of a Lathe

For some power tools such as the grinder, drill press and milling machine, the cutting tool rotates and the cutting speed depends upon the diameter and the speed of the rotating tool. In the case of the lathe the work rotates and the cutting speed depends upon the diameter of the work and the speed at which it rotates.

The cutting speed for work in a lathe is the rate in feet per minute at which the material passes the cutting tool.

Example 1: A cast-iron gear blank 3″ in diameter is being turned in a lathe which is set to run at 75 r.p.m. Calculate the cutting speed.

Solution: The circumference of the work in feet is $\frac{3{\cdot}14 \times 3}{12}$.

The calculated cutting speed in f.p.m. is $\frac{3{\cdot}14 \times 3 \times 75}{12}$

$= 58{\cdot}9.$

Example 2: An aluminum shaft $2\frac{1}{2}''$ in diameter is to be turned in a lathe with a cutting speed of 250 f.p.m. Calculate a suitable speed at which the lathe should be operated.

Solution: Let the speed in r.p.m. be x.

Then, the cutting speed in f.p.m. is $\frac{3{\cdot}14 \times 2{\cdot}5 \times x}{12} = 0{\cdot}654x$.

But, the cutting speed in f.p.m. is 250.

Therefore, $0{\cdot}654x = 250$.

Solving we get, $x = 382$.

The operator will choose a speed as close as possible to 382 r.p.m.

Exercise

B

1. Calculate the cutting speed when a brass rod $1\frac{3}{8}''$ in diameter is turned at a speed of 425 r.p.m.
2. A piece of work $1\frac{1}{4}''$ in diameter is turned at 160 r.p.m. Calculate the cutting speed.
3. A 3″ cylinder of wood is to be turned at 1,000 f.p.m. Calculate the lathe speed in r.p.m.
4. Calculate the lathe speed for turning each of the following:
   (*a*) a $1\frac{3}{4}''$ cast-iron bar at 60 f.p.m.;
   (*b*) a 6″ aluminum wheel at 200 f.p.m.;
   (*c*) a $\frac{3}{4}''$ machine steel piece at 66 f.p.m.
5. If a 4″ mild steel shaft is to be turned at approximately 80 f.p.m., calculate the r.p.m. setting of the lathe.
6. If a $1\frac{1}{2}''$ aluminum rod is to be turned at 300 f.p.m., find a suitable speed.
7. Calculate the r.p.m. setting for turning a cast-iron bar $2\frac{1}{4}''$ in diameter at 50 f.p.m.
8. Calculate the cutting speed of a 3″ gear blank which is being turned in a lathe at 78 r.p.m.
9. What is the speed of a belt driven by an 8″ pulley rotating at 350 r.p.m.?
10. Calculate the r.p.m. at which a lathe should be set to turn each of the following:
    (*a*) 2″ diameter cast iron at 50 f.p.m.;
    (*b*) 2″ diameter cast iron at 60 f.p.m.;
    (*c*) 2″ diameter brass at 150 f.p.m.;
    (*d*) 2″ diameter wood at 1,000 f.p.m.;
    (*e*) 5″ diameter aluminum at 300 f.p.m.

Formula for Cutting Speed and R.P.M.

Let the cutting speed of the rotating part in f.p.m. be S.

Let the diameter of the rotating part in inches be d.

Let the r.p.m. of the turning part be T.

Then, the circumference in feet is $\frac{\pi d}{12}$.

The cutting speed in f.p.m. is $\frac{\pi d}{12} \times T$.

But the cutting speed in f.p.m. is S.

Therefore $$S = \frac{\pi d T}{12}.$$

Explain how this formula may be solved for T to obtain

$$T = \frac{12S}{\pi d}.$$

For any material and cutting tool, equally good results may be obtained for a considerable range of cutting speeds. For most purposes, calculations involving cutting speeds are sufficiently accurate if the value of π is taken as 3.

Hence, $S = \frac{\pi dT}{12}$ simplifies to $S = \frac{dT}{4}$.

Again solving for T we get $T = \frac{4S}{d}$.

Example: Find the approximate r.p.m. of a lathe to give a cutting speed of 55 f.p.m. when a cast-iron piece of material $1\frac{1}{2}''$ in diameter is being turned.

Solution: In this case $S = 55$ and $d = 1{\cdot}5$.

The approximate r.p.m. is $\frac{4 \times 55}{1{\cdot}5} = 147$.

The operator would set the lathe at approximately 147 r.p.m.

Exercise

B

1. Find the cutting speed for each of the following:

(*a*) a 10″ grinding wheel which rotates at 1,530 r.p.m.;

(*b*) a 2″ milling machine cutter which rotates at 105 r.p.m.;

(*c*) a 3″ shaft in a lathe which rotates at 70 r.p.m.;

(*d*) a $\frac{5}{16}''$ drill which rotates at 1,100 r.p.m.

2. Find the r.p.m. for each of the following:

(*a*) a 6″ grinding wheel which is to have a surface speed of 5,000 f.p.m.;

(*b*) a $2\frac{7}{16}''$ cutter on the milling machine which is to cut at a speed of 50 f.p.m.;

(*c*) a $2\frac{1}{2}''$ shaft to be turned at a speed of 300 f.p.m.;
(*d*) a $\frac{3}{8}''$ drill which is to operate at a cutting speed of 110 f.p.m.

3. A grinder rotates at 1,910 r.p.m. Find its diameter if its cutting speed is 7,000 r.p.m.
4. Find the size of a pulley which will have a rim speed of 150 f.p.m. when it is rotated at 104 r.p.m.
5. Find the surface speed of a 6″ grinder operated at a speed of 3,000 r.p.m.
6. An 8″ piece of work is turned in a lathe with a speed of 95 r.p.m. What is the cutting speed?
7. The cutting speeds used for drilling metals in a Canadian motor car plant are listed below:

Material	*Cutting speed in f.p.m.*	*Material*	*Cutting speed in f.p.m.*
Bronze	250	Aluminum	300
Copper	150	Cast iron	80

Calculate the r.p.m. at which a $\frac{3}{4}''$ drill should turn to drill each of the above metals.

8. At what speed should a milling machine be operated if a 6″ cutter is to be used at a cutting speed of 150 f.p.m.?
9. A grinder shaft rotates at a speed of 1,910 r.p.m. What diameter of grinding wheel on this shaft will give a cutting speed of 8,000 f.p.m.?

Feed of a Lathe

The cutting tool of a lathe maintains contact with the work which is turning by means of its horizontal motion. The tool is fed along the work.

The **feed** of the lathe is the distance, in inches, the tool advances for each revolution of the work.

Example 1: By counting, it was found that the work in the lathe turned 120 times while the tool moved $2\frac{1}{2}''$. Calculate the feed.

Solution: The feed in inches per revolution is $\frac{2{\cdot}5}{120} = 0{\cdot}021$.

Example 2: The gears on a lathe are adjusted to give a feed of $\frac{1}{20}''$ when a cut 10″ long is to be taken at 62 r.p.m. Calculate the time required to take a single cut over the work.

Solution: The number of revolutions required for 10″ is 10×20.

The time required in minutes is $\frac{200}{62} = 3{\cdot}2$.

Example 3: A brass rod $2\frac{1}{2}''$ in diameter is to be turned at 120 feet per minute with a feed of 0·020″. If the length of the part machined is 18″, find the time required for one complete cut.

Solution: The number of revolutions for 1 cut is $\frac{18}{0{\cdot}020} = 900$.

The speed of the lathe in r.p.m. is $\frac{4 \times 120}{2{\cdot}5} = 192$.

The time required in minutes is $\frac{900}{192} = 4{\cdot}7$.

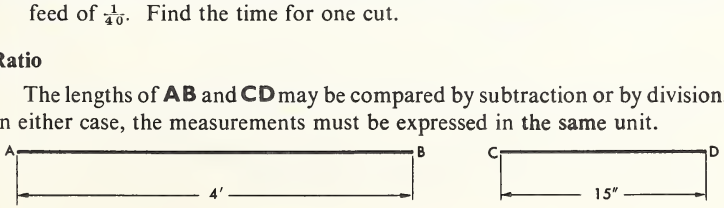
Exercise

B

1. Calculate the feed, in inches per revolution, for each of the following:
   (*a*) the tool advances 4·5″ for 200 revolutions of the work;
   (*b*) the tool advances $2\frac{3}{4}''$ for 242 revolutions of the work;
   (*c*) the tool advances $1\frac{1}{4}''$ for 80 revolutions of the work.
2. Calculate the feed, in inches per revolution, for each of the following:
   (*a*) the tool advances 3″ for 60 revolutions of the work;
   (*b*) the tool advances $2\frac{1}{2}''$ for 80 revolutions of the work;
   (*c*) the tool advances $\frac{7}{8}''$ for 91 revolutions of the work.
3. Find the number of turns the work will make while the tool travels $\frac{3}{4}''$ along a piece of work in a lathe with a feed of $\frac{1}{40}''$.
4. If a feed of $\frac{1}{60}$ is used, how many turns will the work make while the tool travels $6\frac{3}{4}''$?
5. If a lathe is set at a speed of 84 r.p.m. and a cut of 6″ in length requires 4 minutes, find the feed.
6. A piece of work 24″ long is being turned at 80 r.p.m. Find the time required to take one cut over the work if the feed is
   (*a*) $\frac{1}{30}$, (*b*) $\frac{1}{10}$, (*c*) 0·010, (*d*) 0·025.
7. A 3″ cast-iron shaft 24″ long is being turned in a lathe with a cutting speed of 60 f.p.m. and a feed of $\frac{1}{32}''$. Find the time required to take a cut along the work.
8. A 2″ cast-steel rod is being turned for 18″ of its length at 50 f.p.m. with a feed of $\frac{1}{40}$. Find the time for one cut.

Ratio

The lengths of **AB** and **CD** may be compared by subtraction or by division. In either case, the measurements must be expressed in the same unit.



When compared by subtraction, the difference in inches is $48 - 15 = 33$.

When compared by division, the quotient is $\frac{48 \text{ inches}}{15 \text{ inches}} = \frac{16}{5} = 3\frac{1}{5}$.

Such a quotient is called a **ratio.** The quotient in such cases is a number without a unit of measurement. A ratio such as $\frac{16}{5}$ may be written in the form 16:5 and is read "the ratio of 16 to 5".

A ratio is a quotient obtained when two quantities, measured in the same units, are compared by division.

Example: Find the ratio of the areas of two circles, the diameters of which are 6″ and 10″ respectively.

Solution: The area of the 6″ circle in sq. in. is 9π.

The area of the 10″ circle in sq. in. is 25π.

The area of the 6″ circle : the area of 10″ circle $= \frac{9\pi}{25\pi} = \frac{9}{25}$.

It is usual, in stating ratios, to compare the first named quantity with the second.

Exercise

A

1. If the length of **AB** is 4″ and the length of **CD** is 12″, what is the ratio of their lengths?
2. What is the ratio of 440 yards to 1 mile?
3. What is the ratio of two lengths which are 6″ and 1 yard respectively?
4. What is the ratio of two resistances of 100 ohms and 125 ohms?
5. Solder is a mixture containing 40% tin and 60% lead. Find the ratio of the tin to the lead.

B

6. If Bill weighs 156 lb. and Bob weighs 91 lb., find the ratio of their weights.
7. Write down 3 ratios equivalent to each of the following:
   (*a*) 2:3, (*b*) 56:63, (*c*) 39:65.
8. Using only numbers which are multiples of 4 write two ratios equivalent to each of the following:
   (*a*) 5:9, (*b*) 18:21, (*c*) 23:16.
9. Jim's income is \$4,800 per year and his expenses are \$4,400. Joe's income is \$4,200 per year and his expenses are \$4,000. Find
   (*a*) the ratio of their incomes, (*b*) the ratio of their expenses,
   (*c*) the ratio of Jim's savings to Joe's savings.

10. To make a batch of cement a mix is prepared in the following manner: 24 shovelfuls of gravel, 16 of sand and 8 of cement. Express in the simplest form:

(*a*) the ratio of sand to gravel,
(*b*) the ratio of cement to gravel and sand combined,
(*c*) the ratio of sand to cement.

11. Each capsule of a certain Vitamin tablet contains 10,000 i.u. of Vitamin A and 1,750 i.u. of Vitamin D. Find the ratio of Vitamin A to Vitamin D in this capsule.

12. To make liquid milk from dry milk powder we mix 4 tablespoonfuls of powder to $\frac{1}{2}$ pint of water. What is the ratio by volume of the powder to the water?

13. If two gears in mesh have 64 teeth and 120 teeth respectively, find the teeth ratio of the gears.

14. Two pulleys of diameters 4″ and $6\frac{1}{2}$″ respectively are belted together. Find the ratio of their diameters.

15. Two squares have sides of 5″ and 10″ respectively. Find:

(*a*) the ratio of their lengths, (*b*) the ratio of their perimeters,
(*c*) the ratio of their areas, (*d*) the ratio of their diagonals.

16. The first of two cylinders has a diameter of 4″ and a height of 8″, the second has a diameter of 6″ and a height of 12″. Find:

(*a*) the ratio of their diameters, (*b*) the ratio of their heights,
(*c*) the ratio of the areas of their bases, (*d*) the ratio of their volumes.

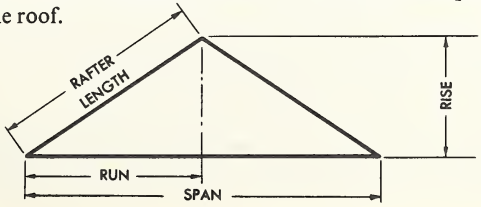
17. A teeter-totter balances when a boy weighing 150 lb. sits 6 feet from the point of support and a boy weighing 100 lb. sits 9 feet from the same point. Find

(*a*) the ratio of the boys' weights,
(*b*) the ratio of their distances from the point of support.

18. With a certain arrangement of ropes and pulleys a man is able to lift 300 lb. by exerting a force of 50 lb. Find the ratio of the load to the pull. This ratio is called the **mechanical advantage** of the machine.

19. By using a screw type of jack, a force of 100 lb. will lift a load of 10 tons. Find the mechanical advantage of this jack.

20. In a gable roof, as illustrated, the ratio of the **rise** to the **span** is called the **pitch** of the roof.



Find the pitch of the roof in each of the following cases:

(*a*) a span of 32 ft. and a rise of 16 ft.,
(*b*) a span of 20 ft. and a rise of 8 ft.,
(*c*) a span of 24 ft. and a rise of 8 ft.,
(*d*) a span of 12 ft. and a rise of 3 ft.,
(*e*) a run of 13 ft. and a rise of 6′ 6″,
(*f*) a rafter length of 10 ft. and a run of 8 ft.,
(*g*) a rafter length of 13 ft. and a run of 12 ft.

21. The **specific gravity** of a substance is the ratio of the weight of the material to the weight of the same volume of water. If water weighs 62·32 lb. per cubic foot, calculate the specific gravity of each of the following substances:

Material	*Weight per cubic foot*	*Material*	*Weight per cubic foot*
cast iron	450	nickel	555
cedar	22	white pine	25
cement	90	concrete	144
ice	58·7	tantalum	1,033
petroleum	55	gold	1,203

22. A rise in temperature of 100° Centigrade is equivalent to a rise in temperature of 180° Fahrenheit. Find the ratio of the lengths of 1° Centigrade division on the temperature scale to 1° Fahrenheit division.

Direct Proportion

Compare the ratio of the times with the ratio of the distances travelled by an aeroplane at a constant speed of 200 m.p.h. after 2 hours and 5 hours respectively.

The ratio of the times is $\frac{2}{5}$.

The distance travelled in 2 hours in miles is 400.

The distance travelled in 5 hours in miles is 1,000.

The ratio of the distances in the same order is $\frac{400}{1{,}000} = \frac{2}{5}$.

Therefore, the ratio of the distances is equal to the ratio of the times. This relationship is a proportion and we may say when the speed is constant the distance travelled is directly proportional to the time.

A **proportion** is the statement that two ratios are equal.

In the proportion above four measures are involved and are related thus 400:1,000 = 2:5. The numbers 400, 1,000, 2 and 5 taken in this order are in

proportion. Also, if the four measures a, b, c and d are known to be in proportion then $a:b = c:d$ or $\frac{a}{b} = \frac{c}{d}$.

Example 1: It is known that the resistance of a conductor of electricity is proportional to its length. If a No. 6 wire has a resistance of 0·403 ohms per 1,000 feet of length, find the resistance for 225 feet of this wire.

Solution: Let the resistance in ohms of 225 ft. of wire be x.

Then the ratio of the resistances is $\frac{x}{0{\cdot}403}$.

The ratio of the lengths is $\frac{225}{1{,}000}$.

Since the resistance is directly proportional to the length

$$\frac{x}{0{\cdot}403} = \frac{225}{1{,}000}.$$

Solving, we find that $x = 0{\cdot}091$.

The resistance of 225 ft. of this wire is 0·091 ohms.

Example 2: Two emery wheels on the same shaft are 10″ and 8″ in diameter respectively. If the grinding speed of the 10″ wheel is 4,800 f.p.m., find the grinding speed of the 8″ wheel.

Solution: It is easily shown that the grinding speeds of wheels turning at the same r.p.m. is directly proportional to the diameters of the wheels.

Let the grinding speed of the 8″ wheel in f.p.m. be x.

The ratio of the grinding speeds is $\frac{4{,}800}{x}$.

The ratio of the diameters in the same order is $\frac{10}{8}$.

Since the speeds are directly proportional to the diameters

$$\frac{4{,}800}{x} = \frac{10}{8}.$$

Solving, we find that $x = 3{,}840$.

Therefore, the grinding speed of the 8″ wheel is 3,840 f.p.m.

Exercise B

Express the relationship in each of the following as a proportion and solve.

1. It is known that a plane has travelled 420 miles in 3 hours. At the same rate, how far will it travel in

(*a*) 5 hours, (*b*) 6 hours, (*c*) 3½ hours, (*d*) 1¼ hours?

2. If the cost of 3 bolts is 25 cents, calculate the cost of 4 dozen of the same kind of bolts.
3. If 6 labourers earn \$264 per week, how many labourers could you hire for \$1,200 per week?
4. Solve the following proportions for the unknown term:

   (*a*) $\frac{x}{4} = \frac{7}{8}$. (*b*) $\frac{15}{80} = \frac{x}{12}$.

   (*c*) $x:2{\cdot}828 = 14:2$. (*d*) $6:8 = 42:x$.

   (*e*) $\frac{6\pi}{10\pi} = \frac{2}{x}$. (*f*) $0{\cdot}04:0{\cdot}4 = 1{\cdot}2:x$.

   (*g*) $\frac{300}{540} = \frac{40}{x}$. (*h*) $1:3 = x:2\frac{1}{4}$.
5. If 3 machines turn out 1,500 pieces of work per week, how many machines would be needed to turn out 2,500 pieces per week?
6. The height of an equilateral triangle is directly proportional to the length of a side. If an equilateral triangle with a side of 4″ has a height of 3·464″, find the heights of the equilateral triangles which have bases as follows:

   (*a*) 8″, (*b*) 2″, (*c*) 5″, (*d*) 10″, (*e*) $\frac{1}{8}$″, (*f*) 0·0833″.
7. Find the length of the sides of each equilateral triangle for which the altitude is as follows: (*a*) 8″, (*b*) 2″, (*c*) 10·392″.
8. The stretch produced in a spring is proportional to the weight attached. If a weight of 15 g. attached to a certain spring produces a stretch of 1·2 cm., find the stretch produced by each of the following weights:

   (i) 10 g., (ii) 32 g., (iii) 7 g.
9. If a weight of 18 g. attached to a spring produces a stretch of 1·5 cm., what weight will be required to stretch the spring

   (i) 6 cm., (ii) 0·8 cm., (iii) 5·3 cm.?
10. It is found that a certain spring is stretched $\frac{1}{2}$ inch when a force of 5 lb. is applied. Find the amount that this spring will be stretched by each of the following loads: (i) 3 lb., (ii) 4 lb., (iii) 7 lb. 8 oz.
11. If a 10-lb. force stretches a spring $\frac{15}{16}$″, find the force required to stretch this spring

    (i) $\frac{3}{4}$″, (ii) $1\frac{1}{4}$″, (iii) $2\frac{1}{4}$″.

Inverse Proportion

Examine the following illustration to find how the ratio of the speeds of two planes is related to the ratio of their travelling times.

Plane **A** travelled a distance of 840 miles in 4 hours but plane **B** took 6 hours to travel the same distance.

The speed of plane **A** in m.p.h. is $\frac{840}{4} = 210$.

The speed of plane **B** in m.p.h. is $\frac{840}{6} = 140$.

The speed of plane **A** : the speed of plane **B** $= \frac{210}{140} = \frac{3}{2}$.

The time of plane **A** : the time of plane **B** $= \frac{4}{6} = \frac{2}{3}$.

If one of these ratios is inverted, then the inverted ratio will be equal to the other.

The speeds of the planes is inversely proportional to their travelling times *or* the travelling times of the planes is inversely proportional to their speeds.

When four measures a, b, c and d are such that the ratio $a:b$ equals the ratio $d:c$, then a and b are inversely proportional to c and d.

Example: When the temperature is kept constant, the volume of a gas is inversely proportional to the pressure to which it is subjected. A volume of 50 cu. ft. of air at a pressure of 20 lb. per sq. in. is compressed into a tank with a volume of 5 cu. ft. Find the new pressure.

Solution: Let the new pressure in lb. per sq. in. be p.

The ratio of the pressures is $\frac{20}{p}$.

The ratio of the volumes taken in the same order is $\frac{50}{5}$.

Since the volume is inversely proportional to the pressure

$$\frac{20}{p} = \frac{5}{50}.$$

Solving, we find that $p = 200$.

Therefore, the pressure in the tank is 200 lb. per sq. in.

Exercise

B

1. The time that a car takes to cover a fixed distance is inversely proportional to its speed. If a car at 30 m.p.h. takes 5 seconds to cover a measured distance, how fast would the car be travelling if it covered the same distance in 3 seconds?
2. The safe load that a beam can carry is inversely proportional to the distance between the supports. If a certain beam can carry a load of 10 tons when the supports are 8 feet apart, what safe load will this beam carry when the supports are placed

   (*a*) 6 feet apart, (*b*) 10 feet apart, (*c*) 20 feet apart?

3. Find the new volume for each of the following:
   (*a*) the pressure on 200 cu. ft. of air at 20 lb. per sq. in. is changed to 90 lb. per sq. in.;
   (*b*) the pressure on 300 cu. ft. of air at 200 lb. per sq. in. is changed to 80 lb. per sq. in.
4. A tank contains 5 cu. ft. of oxygen at a pressure of 200 lb. per sq. in. How many cu. ft. of oxygen at 20 lb. per sq. in. would this provide?
5. At 50 miles per hour, a train takes 24 minutes to make a trip between stations. How long would it take to make the same trip if the speed could be increased to 65 m.p.h.?
6. The number of men laying a gas main is inversely proportional to the length of time that it takes to do the job. If 6 men can lay 1 mile in 20 days, find
   (*a*) how long it will take 15 men to lay one mile;
   (*b*) how many men it would take to lay 1 mile in 2 days.
7. If a, b, c and d are four measures such that a and b are inversely proportional to c and d,
   (*a*) find a when $b = 5$, $c = 6$ and $d = 10$;
   (*b*) find b when $a = 3$, $c = 7$ and $d = 4$;
   (*c*) find d when $a = 5$, $b = 8$ and $c = 10$.
8. The pitch of a note produced by plucking a stretched wire is inversely proportional to the length of the wire. A taut steel wire produces a pitch of 250 vibrations per second when 10 inches of its length vibrates. What will be the measure of the pitch of this wire if
   (*a*) only 5″ of it is allowed to vibrate,
   (*b*) the vibrating portion is 12″ long,
   (*c*) only 8″ is allowed to vibrate?
9. The pitch of a note produced by plucking a stretched string is directly proportional to the tension of the string. If a certain string gives a pitch of 200 v.p.s. when it is under a tension of 20 lb., what will be the pitch when the tension is
   (*a*) 25 lb., (*b*) 15 lb., (*c*) 40 lb.?

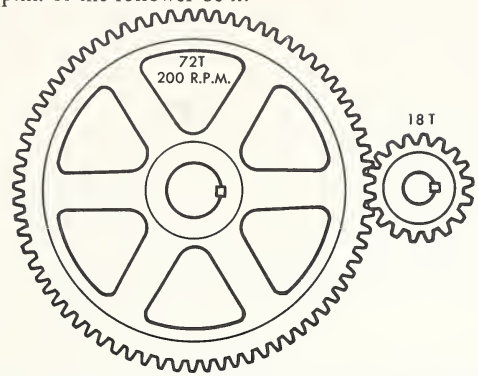
Gear Speeds

Power from one rotating shaft to another can be transmitted by means of gears. The gear on the power shaft is called the **driver** and that on the other shaft, the **follower**.

The following example will illustrate the relationship between the r.p.m. of gears in mesh and the number of teeth on the gears respectively.

A driver with 72 teeth meshes with a follower of 18 teeth. If the r.p.m. of the driver is 200, find the r.p.m. of the follower.

Let the r.p.m. of the follower be x.



The number of teeth of the driver passing the point of contact in 1 minute is 72×200.

The number of teeth of the follower passing the point of contact in 1 minute is $18 \times x$.

Therefore, $18x = 72 \times 200.$

Solving, we find that $x = 800.$

Therefore, the r.p.m. of the follower is 800.

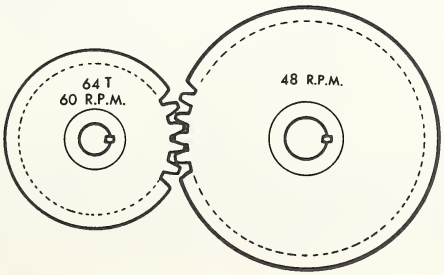
In this case, the driver-follower teeth ratio is $\frac{72}{18} = \frac{4}{1}$.

The driver-follower r.p.m. ratio is $\frac{200}{800} = \frac{1}{4}$.

This would indicate that when *two gears are in mesh, their speeds are inversely proportional to the numbers of their teeth.* It can be shown that this relationship is always true and may be expressed as follows:

$$\frac{\textbf{R.P.M. of the follower}}{\textbf{R.P.M. of the driver}} = \frac{\textbf{Number of teeth on the driver}}{\textbf{Number of teeth on the follower}}.$$

Example: If a driving gear has 64 teeth and turns at 60 r.p.m., select a follower to give a speed of 48 r.p.m.



Solution:
Let the number of teeth on the follower be t.

The driver-follower teeth ratio is $\frac{64}{t}$.

The driver-follower speed ratio is $\frac{60}{48}$.

Therefore, $\frac{64}{t} = \frac{48}{60}$. Note that one of these ratios has been inverted.

Solving, we find that $t = 80$.

Select a gear of 80 teeth for the follower.

Gear Progressions

In the shop where a variety of speed ratios have to be made by using gears, standard sets are provided. One such set would contain gears with 20, 24, 28, 32, ... 120 teeth respectively. This is a gear progression of 4.

Example: A shaft that must rotate at 600 r.p.m. is connected by gears to a drive shaft which rotates at 400 r.p.m. Select two suitable gears and indicate the shaft on which each must be placed.

Solution: The driver-follower speed ratio is $\frac{400}{600} = \frac{2}{3}$.

Therefore, $\frac{\text{number of teeth on the driver}}{\text{number of teeth on the follower}} = \frac{3}{2}$.

This ratio may be expressed by any fraction equivalent to $\frac{3}{2}$.

Therefore, $\frac{\text{number of teeth on the driver}}{\text{number of teeth on the follower}} = \frac{24}{16} = \frac{30}{20} = \frac{36}{24} = \frac{39}{26} = \text{etc.}$

If a gear progression of 4 is available, the required speed ratio may be obtained by any one of the following pairs of gears placed as follows:

Number of teeth on the driver....	36	48	60	72	84	96	108	120
Number of teeth on the follower...	24	32	40	48	56	64	72	80

Exercise
B

1. A power shaft rotating at 200 r.p.m. is fitted with a gear of 36 teeth. What size of gear should be placed on the follower shaft to give a speed of (*a*) 50 r.p.m., (*b*) 80 r.p.m., (*c*) 300 r.p.m., (*d*) 180 r.p.m.?

2. A drive shaft rotates with a speed of 500 r.p.m. Find the speed with which a follower shaft will rotate when the number of teeth on the driver gear and the number of teeth on the follower gear respectively are as follows:

   (*a*) 80 and 120, (*b*) 60 and 40, (*c*) 20 and 84, (*d*) 24 and 70.

3. A gear with 50 teeth, rotating at 300 r.p.m. in a clockwise direction, drives a gear with 84 teeth. What will be the speed and direction of rotation of the follower?

4. A gear with 120 teeth, rotating in a counter-clockwise direction with a speed of 74 r.p.m., drives an idler with 24 teeth which in turn drives a third gear with 40 teeth. What will be the speed and direction of rotation of the gear with 40 teeth?

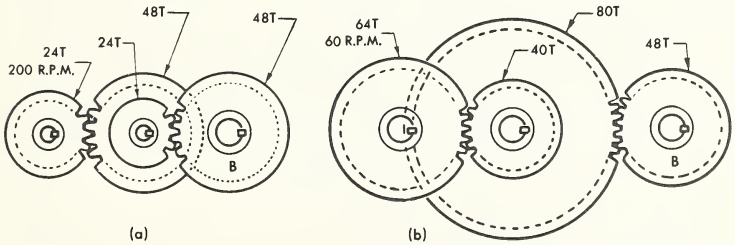
5. In what way does an idler, placed between the driver and the follower, affect the rotation of the follower?

6. The stud gear of a lathe rotates at 200 r.p.m. From a gear progression of 4, select suitable gears, one of which is to be placed on the stud, the other on the lead screw so that the lead screw has an r.p.m. of 150.

7. A drive shaft with a speed of 500 r.p.m. is geared to a second shaft which is required to have a speed of 850 r.p.m. If a gear progression of 4 is available, select a suitable gear for each shaft.

8. A drive shaft rotating at 200 r.p.m. has a gear with 48 teeth in mesh with a gear of 56 teeth on one end of a second shaft. On the other end of the second shaft is a gear with 80 teeth in mesh with a gear of 60 teeth on a third shaft. What is the speed at which the third shaft rotates? Such an arrangement of gears is called a **compound gear train.**

9. In the following compound gear train, determine the speed of gear **B.**

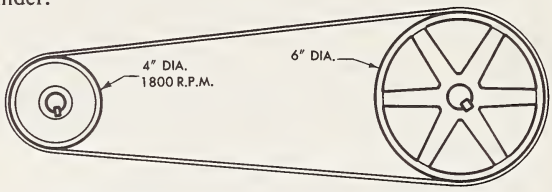


Pulley Speeds

When power is transmitted by means of a belt or a chain, the speed of the follower shaft may be regulated by the sizes of the pulleys.

The following example will show the relationship between the speeds of two belted pulleys and their diameters.

A motor rotating at 1,800 r.p.m. is belted to a grinder. If the pulley on the motor is 4″ and the pulley on the grinder is 6″ in diameter, find the r.p.m. of the grinder.



Motor Pulley **Grinder Pulley**

Let the speed of the grinder in r.p.m. be x.

Then the belt speed of the motor in f.p.m. is $\frac{\pi \times 4 \times 1{,}800}{12}$.

Then the belt speed at the grinder in f.p.m. is $\frac{\pi \times 6 \times x}{12}$.

But the belt speed must be the same throughout.

Therefore, $$\frac{4 \times \pi \times 1{,}800}{12} = \frac{6 \times \pi \times x}{12}.$$

Solving we find that $x = 1{,}200.$

The speed of the grinder is 1,200 r.p.m.

In this case, the driver-follower diameter ratio is $\frac{4}{6} = \frac{2}{3}$.

The driver-follower speed ratio is $\frac{1{,}800}{1{,}200} = \frac{3}{2}$.

This would indicate that *the speeds of two belted pulleys are inversely proportional to their diameters.* It can be shown that this relationship is always true and may be expressed as follows:

$$\frac{\textbf{R.P.M. of follower pulley}}{\textbf{R.P.M. of driver pulley}} = \frac{\textbf{Diameter of driver pulley}}{\textbf{Diameter of follower pulley}}.$$

Example: A 4″ pulley turning at 200 r.p.m. drives a 10″ pulley. What is the speed of the 10″ pulley?

Solution: Let the r.p.m. of the 10″ pulley be x.

The diameter ratio is 4:10.

The speed ratio in the same order is 200:x.

Therefore, $$\frac{x}{200} = \frac{4}{10}.$$

Note that one of these ratios has been inverted.

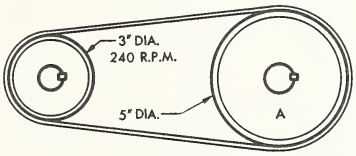
Solving we find that $x = 80.$

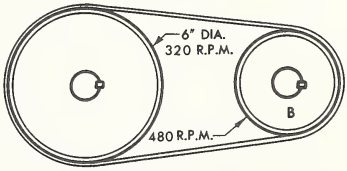
Therefore, the r.p.m. of the 10″ pulley is 80.

Exercise

B

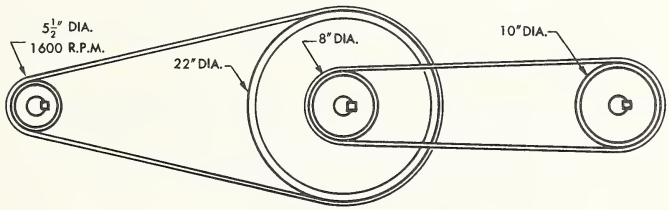
1. A ventilating fan with a 14″ pulley is driven by a motor with an 8″ drive pulley. If the motor speed is 1,680 r.p.m. find the speed of the fan.
2. A pulley 8″ in diameter turning at 120 r.p.m. is cross-belted to a pulley 12″ in diameter. Find the speed of the 12″ pulley.
3. A shaft, with a 10″ pulley attached, is rotating at 360 r.p.m. and is to be belted to another shaft which must rotate at 480 r.p.m. Find the size of the pulley for the second shaft.
4. A line shaft pulley of 8″ is rotating at 1,200 r.p.m. If this shaft is to be belted to another one which must run at 1,000 r.p.m., what size pulley should be used?
5. (*a*) Find the r.p.m. of pulley **A**. (*b*) Find the diameter of pulley **B**.



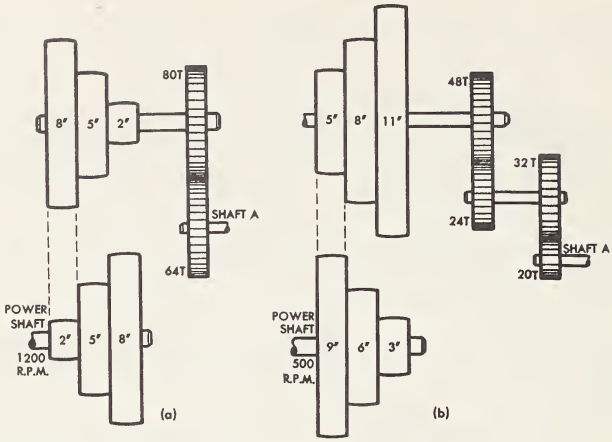


6. Find the sizes of three sets of pulleys to transmit motion from a driving shaft at 80 r.p.m. to a follower shaft at 60 r.p.m.

   State on which shaft each pulley will be placed.
7. Calculate the r.p.m. of the 10″ pulley in the set of compound pulleys illustrated below.



8. A set of three cone pulleys of diameters 6″, 4″ and 2½″ are attached to a wood lathe. A similar set in reverse order are attached to the motor shaft which drives the lathe. If the motor rotates at 1,400 r.p.m., find the three possible speeds for the lathe shaft.
9. Find the three speeds of the shaft **A** which can be obtained by means of each of the cone pulley systems sketched on the next page.



10. A grinder, driven by a belt from a line shaft, operates at 1,200 r.p.m. when fitted with a 4″ pulley.

(*a*) If the 4″ pulley on the grinder is replaced by a 3″ pulley, at what speed will the grinder operate?

(*b*) What size of pulley should be placed on the grinder shaft if a speed of 480 r.p.m. is required?

Taper Measurements

The end of a neatly sharpened pencil is an example of a taper. If the large diameter is $\frac{1}{4}''$ and the diameter at the end of the lead is $\frac{1}{32}''$ then the amount of the taper in inches is $\frac{1}{4} - \frac{1}{32} = \frac{7}{32}$.

The amount of **taper** is the difference in the end diameters of the tapered portion.

It is usual, however, to designate the measurement of taper in terms of the taper per foot of length.

Example: The end diameters of a tapered piece of work are $2\frac{1}{2}''$ and $3\frac{1}{4}''$ respectively. If the material is 8″ in length, find the taper per foot.

Solution 1: The taper for 8″, in inches, is $3\frac{1}{4} - 2\frac{1}{2} = \frac{3}{4}$.

The taper per inch, in inches, is $\frac{3}{4} \div 8 = \frac{3}{32}$.

The taper per foot, in inches, is $\frac{3}{32} \times 12 = 1\frac{1}{8}$.

Solution 2: Let the taper per foot, in inches, be x.

Then, $$\frac{x}{\frac{3}{4}} = \frac{12}{8}.$$

Solving, we find that $x = 1\frac{1}{8}$.

Therefore, the taper is $1\frac{1}{8}$ inches per foot.

Standard Tapers

1. Brown and Sharpe Standard tapers are $\frac{1}{2}$ inch per foot, except for No. 10 which is 0·5161 inch per foot.

2. Morse Standard tapers vary with the different numbers but all are nearly $\frac{5}{8}$ inch per foot.

3. Jarno tapers are all 0·6 inch per foot.

4. Standard $\frac{3}{4}$ inch tapers are all $\frac{3}{4}$ inch per foot.

5. British Standard taper pins have a taper of $\frac{1}{4}$ inch per foot.

For detailed information regarding taper series refer to a handbook.

Example: A No. 7 Brown and Sharpe standard taper has a small diameter of 0·600″ and is $2\frac{7}{8}$″ long. Calculate the large diameter.

Solution: The taper per foot in inches is 0·500.

The taper per inch, in inches, is $\frac{0{\cdot}500}{12}$.

The taper for $2\frac{7}{8}$″, in inches, is $\frac{0{\cdot}500}{12} \times 2{\cdot}875$

$= 0{\cdot}1198.$

The large diameter, in inches, is $0{\cdot}600 + 0{\cdot}120$

$= 0{\cdot}720.$

Exercise

B

1. If the taper per inch is 0·050″, find the taper per foot.

2. If the taper per inch is 0·042″, find the taper per foot.

3. If the taper per foot is 0·623″, find the taper per inch.

4. If the taper for 5″ of work is 0·480″, find the taper per foot.

5. Find the difference in diameters 3″ apart on a steep taper of $3\frac{1}{2}$″ per foot. If the large diameter is 2″, find the small diameter.

6. Find the difference in diameters 8″ apart on a taper of $\frac{3}{4}$″ per foot. If the small diameter is 1·516″, find the large diameter.

7. A plug $7\frac{1}{4}$″ long has a Morse No. 6 taper of 0·626″ per foot. If the large diameter is 2·494″, what is the small diameter?

8. The small diameter of a piece of work with a No. 12 Brown and Sharpe taper is 1·5000″. Find the large diameter if the work is $7\frac{1}{8}$″ long.

9. For each of the following standard Morse tapers, calculate the taper per foot.

Taper Number	*Small diameter in inches*	*Large diameter in inches*	*Length in inches*
0	0·252	0·356	2
1	0·369	0·475	$2\frac{1}{8}$
2	0·572	0·700	$2\frac{9}{16}$
3	0·778	0·938	$3\frac{3}{16}$
4	1·020	1·231	$4\frac{1}{16}$
5	1·475	1·748	$5\frac{3}{16}$

10. For each of the following standard Jarno tapers, calculate the large diameter.

Taper Number	*Small diameter in inches*	*Length in inches*
1	0·1	$\frac{1}{2}$
2	0·2	1
5	0·5	2·5
n	$\frac{n}{10}$	$\frac{n}{2}$

11. For each of the following standard Brown and Sharpe tapers, calculate the large diameter.

Taper Number	*Small diameter in inches*	*Length in inches*
1	0·200	$\frac{15}{16}$
2	0·250	$1\frac{3}{16}$
3	0·312	$1\frac{1}{2}$
4	0·350	$1\frac{11}{16}$
5	0·450	$2\frac{1}{8}$
10	1·0446	5
15	2·25	$8\frac{3}{4}$

Taper Turning

Tapers on round stock are usually machined on a lathe by one of three methods.

1. Offsetting the tailstock; 2. Using a taper attachment; 3. Using the compound rest.

When the taper is not too steep, the uniformly decreasing diameter of the work to be tapered in the lathe may be obtained by moving the tailstock a slight distance off the centre line. The amount moved is called the **offset.**

Assume that the work has already been turned to the size of the large diameter, then the diameter of the other end must be reduced by an amount equal to the difference in the end diameters. The additional depth of cut at the small end equals $\frac{1}{2}$ the difference in the end diameters. By moving the tailstock forward this amount and operating the cutting tool in the usual manner, the required taper will be cut.

Example 1: Find the offset of a tailstock necessary to cut a taper of $\frac{3}{4}$ inch per foot on a piece of work 10″ long.

Solution: The difference in end diameters in inches is $\frac{\frac{3}{4} \times 10}{12} = \frac{5}{8}$.

Therefore, the offset in inches is $\frac{1}{2} \times \frac{5}{8} = \frac{5}{16}$.

Example 2: Calculate the offset of the tailstock required to turn a taper of $\frac{1}{4}$ inch per foot for a distance of 5 inches on a piece of work 16″ long.

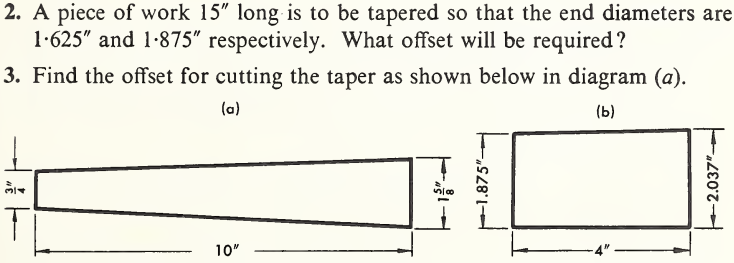
Solution: Since the work is held between centres at each end, assume that the taper is to extend for the entire length of the work.

The taper in inches for the entire length would be $\frac{1}{4} \times \frac{16}{12} = \frac{1}{3}$.

The offset in inches is $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.

Exercise

1. Calculate the offset of the tailstock which will be required to cut a taper of $\frac{1}{2}$ inch per foot over the entire length of a piece of material 8 inches long.
2. A piece of work 15″ long is to be tapered so that the end diameters are 1·625″ and 1·875″ respectively. What offset will be required?
3. Find the offset for cutting the taper as shown below in diagram (*a*).



4. Find the offset for cutting the taper shown in diagram (*b*).
5. Find the offset for turning a taper 10″ long on a bar 14″ long when the taper per foot is 0·600″.
6. On a bar 18″ long, a piece 16″ long is to be tapered. If the end diameters are 1″ and $1\frac{5}{8}$″, find the offset.
7. A taper of $\frac{5}{8}$″ per foot is to be cut over 6 inches on a piece of work 14″ long. Calculate
   (*a*) the offset,
   (*b*) the diameter of the small end if the larger diameter is 1·25″.

8. Find the offset required to turn a taper for $9\frac{3}{4}''$ on a piece of work $10\frac{7}{8}''$ in length if the small diameter is 1·648″ and the large diameter is 1·750″.
9. A piece of work 18″ in length with a large diameter of 2·350″ is to have a taper cut for a distance of $14\frac{3}{4}''$. If the smaller diameter is 2·075″, find the offset of the tailstock.
10. A tailstock on a lathe is offset $\frac{3}{16}$ in. and a taper is cut on a piece of work 8″ long. What is the taper per foot on this work?
11. A piece of work 18 inches long is to be tapered from the centre to each end. On one end the taper is to be $\frac{1}{2}$ inch per foot and the other half is to have a taper of $\frac{3}{4}$ inch per foot. Find the offset for each.
12. A taper is cut the entire length of a piece of work 10 inches long with an offset of $\frac{5}{16}$ inches. What will be the offset to cut the same taper per foot on a piece of work 8 inches long?

Review Exercise

1. Calculate the rim speed of a grinder which is 7″ in diameter and turns at 900 r.p.m.
2. Find the time required to drill a hole $\frac{1}{2}''$ in diameter 6″ deep if the feed of the drill is 0·007″ and the r.p.m. is 800.
3. At what r.p.m. should a 4″ cutter be operated in a milling machine if the cutting speed is not to exceed 250 f.p.m.?
4. What is the speed of a belt driven by a 4″ pulley at 1,400 r.p.m.?
5. Find the approximate r.p.m. at which to set a lathe to give a cutting speed of 60 f.p.m. when the diameter of the material is $2\frac{1}{4}''$.
6. If a lathe has a feed of $\frac{1}{48}$ inch and is turning at 56 r.p.m., calculate the time required to take three cuts over a piece of work 14 inches long. Allow 1 minute for adjustments.
7. A square has a side of 8 inches. Calculate the length of the diagonal. State in simplest form the ratio of the length of the side to the length of the diagonal.
8. A square has a side of 12 inches. Find the ratio of the length of the side to the length of the diagonal.
9. If a certain piece of wire has a resistance of 0·56 ohms and is 125 feet long, find the resistance of 300 feet of the same kind of wire.
10. Solve the equation $\dfrac{36{\cdot}4}{28{\cdot}4} = \dfrac{86{\cdot}5}{x}$.
11. If two gears with 84 teeth and 36 teeth are in mesh and the gear with 84 teeth is turning at 240 r.p.m., find the speed of the 36 toothed gear.
12. A 6″ diameter pulley is turning at 1,320 r.p.m. and is belted to a 10″ pulley. Find the speed of the 10″ pulley.

13. If the taper for a 5″ length is $\frac{1}{4}$″, what is the taper for 12″?
14. If the taper is 0·6 inches per foot, what is the taper for $6\frac{1}{2}$″?
15. Find the offset for cutting a taper of 0·6″ per foot over a piece of work 8″ between centres.
16. A piece of work 14″ long is to be tapered for 6″ of its length. If the small diameter is $2\frac{1}{2}$″ and the large diameter is $2\frac{3}{4}$″, calculate the offset.

CHAPTER 19

Geometrical Constructions and Relationships

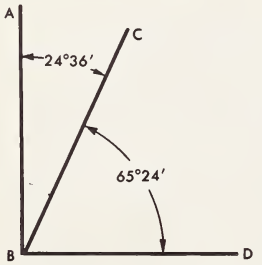
Angles

An **angle** is the amount of turning of a line about a fixed point.

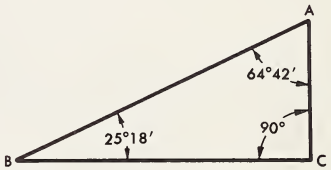
An **angle** may be considered to be the difference in direction between two straight lines.

Adjacent angles are two angles with the same vertex and on opposite sides of a common arm.

Complementary angles are two angles which have a sum of 90°.

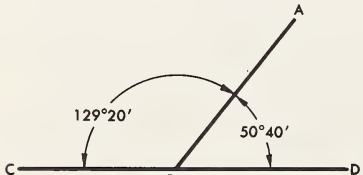


∠**ABC** and ∠**CBD** are adjacent complementary angles.

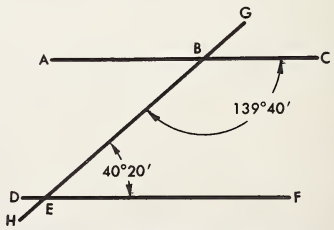


∠**BAC** and ∠**ABC** are complementary angles.

Supplementary angles are two angles which have a sum of 180°.

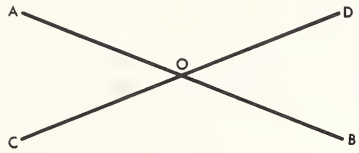


∠**ABD** and ∠**ABC** are supplementary angles.



∠**CBE** and ∠**BEF** are supplementary angles.

When two straight lines intersect, two pairs of **vertically opposite angles** are formed.



$\angle$**AOD** and $\angle$**COB** are vertically opposite angles.
$\angle$**DOB** and $\angle$**AOC** are vertically opposite angles.

Exercise

A

1. What is the complement of 60°, 45°, 80°, 89° 30′?
2. What is the supplement of 70°, 80°, 120°, 20°, 60° 30′?
3. If two angles of a triangle are 55° and 45°, what is the size of the third angle?
4. If the following angles are bisected, state the size of each part: 70°, 118°, 46° 30′, 18° 24′ 36″, 16° 27′, 37° 18′.
5. If $\angle \mathbf{A} = 36°$ and $\angle \mathbf{B} = 24°$, what is the value of $\frac{\mathbf{A}+\mathbf{B}}{2}$ and $\frac{\mathbf{A}-\mathbf{B}}{2}$?
6. The diagonals of a rectangle intersect at 50°. State the sizes of the other three angles at the intersection point.
7. A ship sails south from latitude 50° 28′ to latitude 50° 24′. State the distance that it has travelled in nautical miles.
8. A ship sails north from the equator to latitude 2°. State the distance that it has travelled in nautical miles.
9. (*a*) What size of angle is equal to its complement?
   (*b*) What size of angle is equal to its supplement?
   (*c*) Express in terms of x the complement of $x°$; complement of $45 + \frac{x}{2}$.
   (*d*) Express in terms of y the supplement of $y°$; supplement of $90 - y$.
10. Two straight lines intersect. One of the angles formed is $a°$ and its adjacent angle is $b°$. State the relationship between a and b.

B

11. If two angles are 25° 46′ and 35° 22′ respectively, find
    (*a*) their sum, (*b*) their difference.
12. If two angles of a triangle are 50° 54′ and 80° 25′, find the third angle.
13. The sum of two adjacent angles is 100°. If the smaller one is 25° 18′, find the larger angle.

14. Calculate the complement of each of the following angles:
(*a*) 37°, (*b*) 13° 45′, (*c*) 85° 30′, (*d*) 80° 10′ 30″, (*e*) 10° 43′, (*f*) 9° 45′ 45″.

15. If each of the following angles were bisected, state the size of each of the angles formed:
(*a*) 45°, (*b*) 10° 30′, (*c*) 15° 40′, (*d*) 87° 15′, (*e*) 16° 30′ 40″, (*f*) 130° 25′ 18″.

16. If angle $\mathbf{A} = 15° 18'$ and angle $\mathbf{B} = 50° 6'$, find the value of angle $\frac{\mathbf{B} + \mathbf{A}}{2}$ and of angle $\frac{\mathbf{B} - \mathbf{A}}{2}$.

17. Calculate the supplement of each of the following angles:
(*a*) 135°, (*b*) 82° 10′, (*c*) 170° 20′, (*d*) 35° 46′, (*e*) 165° 28′ 10″, (*f*) 55° 47′ 59″.

18. Draw four sets of pairs of intersecting lines. In each case, measure each pair of vertically opposite angles. State a general conclusion.

19. Two straight lines **AB** and **CD** intersect at **E**. The angle **AEC** is 58° 33′. Find the size of each of the other three angles.

20. One of the angles formed by the diagonals of a rectangle is 28° 45′. Find by calculation the size of each of the other three angles formed by the diagonals.

21. Two straight lines intersect each other. If one angle so formed is $x°$ and the angle adjacent to it is $11x°$, calculate the numerical measure of each angle.

22. If **AB** intersects **CD** at **O** and $\angle\mathbf{AOC}$ is 38° 45′ 30″, calculate
(*a*) the size of each of the other angles.
(*b*) the sum of $\frac{1}{2}\angle\mathbf{AOC}$ and $\frac{1}{2}\angle\mathbf{BOC}$.

23. If three lines are drawn from a point to form three angles two of which are 87° 35′ 24″ and 135° 28′ 47″, find the size of the third angle.

24. A ship sails south from north latitude 50° 24′ to north latitude 48° 39′. Calculate the distance it travels in nautical miles.

25. If a ship sails south from north latitude 2° 38′ to south latitude 8° 29′, find the distance it travels in nautical miles.

26. A ship starts at north latitude 35° 25′ and steams north at 20 knots for 8 hours. Calculate its new latitude.

27. If an aeroplane travels due south from north latitude 4° 33′ to south latitude 5° 42′ in 3 hours, find
(*a*) its average speed in knots,
(*b*) its average speed in statute miles per hour.

Properties of Triangles

The following properties of triangles or relationships between the measures of a triangle have been developed.

1. All the measurements of a triangle can be determined when any one of the following sets of measurements is known:

(*a*) the lengths of the three sides;

(*b*) the lengths of two sides and the size of the angle contained by these sides;

(*c*) the length of one side and the sizes of two of the angles.

2. The sum of the three angles of a triangle is 180°.

3. The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

4. If two sides of a triangle are equal, the angles opposite these sides are equal.

5. Each angle of an equilateral triangle is 60°.

6. The altitude drawn from the vertex of an isosceles triangle to the base bisects the base.

Some additional properties of triangles are illustrated by the following exercise.

Exercise

B

1. Construct triangle **ABC** with **AB** $= 3''$, **BC** $= 4''$ and **CA** $= 3\frac{1}{2}''$. Construct from each vertex the altitude to the opposite side.

2. Construct any triangle. Draw the three altitudes of this triangle. State a conclusion with respect to the altitudes of a triangle.

3. Without using a protractor, construct triangle **ABC** with $\angle$**ABC** $= 60°$, **AB** $= 2''$ and **BC** $= 3''$.

(*a*) Measure and record the length of **AC**.

(*b*) Draw a median from **A** to **BC**.

A **median** of a triangle is a line drawn from a vertex to the midpoint of the opposite side.

(*c*) Draw the medians from **B** to **AC** and from **C** to **AB**.

4. Draw any triangle. Construct the three medians of this triangle. State a conclusion with respect to the medians of a triangle.

5. Draw triangle **ABC** with **AB** $= 3\frac{3}{4}''$, **AC** $= 4\frac{1}{4}''$ and **BC** $= 3\frac{1}{2}''$. Draw **AD** the median from **A** and **BE** the median from **B** to intersect **AD** at **G**. Bisect **AG** at **X**. Compare with your compasses the lengths of **AX**, **XG** and **GD**. Bisect **BG** at **Y**. Compare the lengths of **BY**, **YG** and **GE**.

6. Draw any triangle **KMN** with its three medians. By measuring, check the relationship that the medians divide each other in the ratio of 2 : 1.

7. Construct, without a protractor, the triangle **ABC** with $\angle$**BAC** = 60°, $\angle$**ABC** = 45° and **BC** = $4\frac{3}{8}$".
   (*a*) Draw the bisector of $\angle$**A** and of $\angle$**B** respectively to intersect at **I**.
   (*b*) From **I** construct the perpendiculars to each of the three sides of the triangle. Measure and compare the lengths of these perpendiculars.
   (*c*) Draw any triangle and repeat this construction.
   (*d*) State a relationship indicated by your drawings.

Additional Properties of Triangles

1. The three altitudes of a triangle meet at a point. This point is called the **ortho-centre** of the triangle.

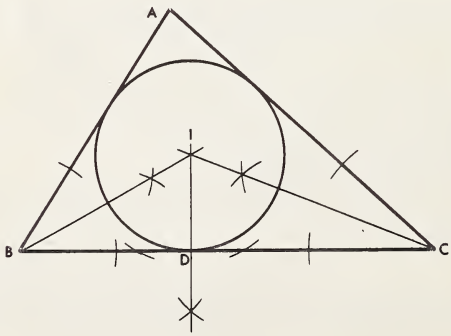
2. The three medians of a triangle meet at a point which is called the **centroid** of the triangle. The centroid is two-thirds of the distance from a vertex to the mid-point of the opposite side.

3. The point of intersection of the bisectors of any two angles of a triangle is equidistant from the three sides.

To Inscribe a Circle in a Triangle

Let **ABC** be a given triangle.

It is required to inscribe a circle in the triangle **ABC.**



Method: Bisect $\angle$**ABC** and $\angle$**ACB** and produce the bisectors to meet at **I.**

Construct a perpendicular from **I** to **BC** to meet **BC** at **D.**

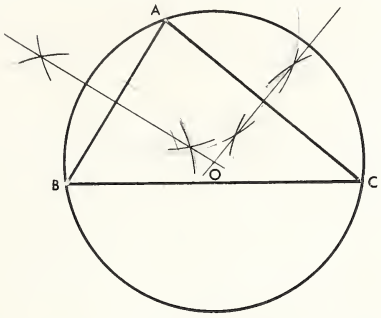
With centre **I** and radius **ID**, construct a circle.

This is the inscribed circle of the triangle **ABC.**

To Circumscribe a Circle about a Triangle

Let **ABC** be a given triangle.

It is required to circumscribe a circle about triangle **ABC**.



Method: Construct the right bisectors of **AB** and **BC** and produce them to meet at **O**.

With centre **O** and radius **OA**, construct a circle.

This is the required circum-circle.

Exercise

A

1. Define
   (*a*) median, (*b*) centroid, (*c*) circum-circle,
   (*d*) inscribed circle, (*e*) hypotenuse.
2. Explain how to circumscribe a circle about a triangle.
3. Explain how to inscribe a circle in a triangle.
4. Under what conditions would the in-centre and the circum-centre of a triangle be the same point?

B

5. Draw a triangle with length of sides 2″, 2½″ and 3″ respectively. Inscribe a circle in this triangle. Measure and record the radius of the inscribed circle.
6. Draw a triangle with the length of two of its sides 3″ and 4″ respectively and the contained angle 60°. Inscribe a circle in this triangle. Measure and record its radius.
7. Construct triangle **ABC** with **BC** = 2½″, $\angle$**B** = 120° and $\angle$**C** = 30°. Inscribe a circle in this triangle. Measure and record its radius.
8. Construct a triangle with the lengths of its sides 3″, 4″ and 5″ respectively. Inscribe a circle in this triangle. Measure and record its radius.

9. Construct a triangle with two sides 2″ and 3″ and the contained angle 45°. Circumscribe a circle about this triangle. Measure and record the radius of the circle.

10. Draw a triangle with one side 4″, the angle opposite this side 30° and one of the angles adjacent to this side 90°. Circumscribe a circle about this triangle. Note the position of the centre of the circum-circle of this right-angled triangle.

11. Draw any right-angled triangle. What is the position of its circum-centre?

12. Construct a circle about a triangle which has two sides of 3″ and $3\frac{1}{2}$″ and a contained angle of 120°. Measure and record the length of the radius of the circum-circle.

13. State the location of the circum-centre of a triangle if it is
(*a*) acute-angled, (*b*) right-angled, (*c*) obtuse-angled.

14. Draw a semicircle on any line **AB** as diameter.
(*a*) Select any point **C** on the circumference. Join **AC** and **BC**. Measure $\angle$**ACB**.
(*b*) Select at least one more point **K** on the circumference. Construct $\angle$**AKB**. Measure and record its size.

An **angle in a semicircle** is one with its vertex on the circumference and its arms passing through the ends of the diameter of the semicircle.

15. Draw any three angles in a semicircle. Measure and record the size of each. State a conclusion about the size of angles in a semicircle.

Additional Properties of Triangles

1. The mid-point of the hypotenuse of a right-angled triangle is equidistant from the three vertices.

2. An angle in a semicircle is a right angle.

Congruency

Geometric figures which are equal in every respect are **congruent** to each other.

Exercise

B

1. Draw any triangle **ABC**.
(*a*) Construct a triangle **DEF** such that **DE** = **AB**, **EF** = **BC** and **DF** = **AC**.
(*b*) Compare the size of each of the angles of triangle **DEF** with that of the corresponding angle of triangle **ABC**.
(*c*) Draw the altitude from **A** to **BC** and the altitude from **D** to **EF**. Measure and compare these altitudes.

(*d*) How does the area of triangle **ABC** compare with the area of triangle **EDF**?

What conclusion about these two triangles is suggested by your comparison?

2. Draw any triangle **XYZ**.
   (*a*) Make a triangle **MNK** such that **MN** = **XY**, $\angle$**N** = $\angle$**Y** and **NK** = **YZ**.
   (*b*) Compare the sizes of the remaining parts of these two triangles.
   (*c*) Draw the altitudes from **X** and **M** respectively to the opposite sides.
   (*d*) Compare the areas of the two triangles.

   What conclusion about these triangles is suggested by your comparison?
3. Draw any triangle **RST**.
   (*a*) Make a triangle **UVW** such that **VW** = **ST**, $\angle$**V** = $\angle$**S** and $\angle$**W** = $\angle$**T**.
   (*b*) Compare the size of the remaining principal parts of these triangles.
   (*c*) Draw the altitudes from **R** and **U** respectively.
   (*d*) Compare the areas of these triangles.

   What conclusion about these triangles is suggested by your comparison?
4. Draw any triangle **ABC** in which $\angle$**B** is acute and **AB** is longer than **BC**.
   (*a*) Construct triangle **MNK** such that $\angle$**N** = $\angle$**B**, **NM** = **BA** and **MK** = **AC**.
   (*b*) Compare the principal parts of these triangles.

   Examine these triangles very carefully before suggesting any conclusion.
5. Draw any triangle **DEF** with **DE** = 1″, **EF** = $1\frac{1}{2}$″ and **DF** = 2″.
   (*a*) Make at least two different triangles **GHK** and **MNP** such that $\angle$**M** = $\angle$**G** = $\angle$**D**, $\angle$**N** = $\angle$**H** = $\angle$**E** and $\angle$**P** = $\angle$**K** = $\angle$**F**.
   (*b*) Measure the lengths of the sides of these triangles.
   (*c*) Compare the ratios of corresponding sides.

Additional Properties of Triangles

If three sides of one triangle are respectively equal to three sides of another triangle the triangles are congruent.

If two sides and the contained angle of one triangle are respectively equal to two sides and the contained angle of another triangle, the triangles are congruent.

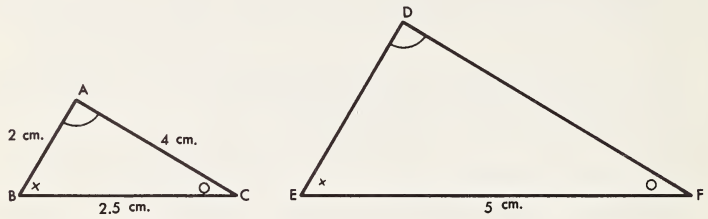
If two angles and a corresponding side of two triangles are respectively equal the triangles are congruent.

Similar Triangles

Any two triangles, which have the angles of one respectively equal to the angles of the other and the ratios of corresponding sides equal, are **similar triangles.**

It has been found that if two triangles have the angles of one respectively equal to the angles of the other the triangles will be similar.

Example: **ABC** and **DEF** are two similar triangles with dimensions as indicated.



Find the length of **DE** and **DF**.

Since the angles of triangle **ABC** are respectively equal to the angles of **DEF**, then the ratios of corresponding sides are equal.

Let the length of **DE** in centimetres be x.

then $\frac{x}{5} = \frac{2}{2{\cdot}5}$

and $x = 4$

Then **DE** is 4 cm.

Let the length of **DF** in centimetres be y.

then $\frac{y}{4} = \frac{5}{2{\cdot}5}$

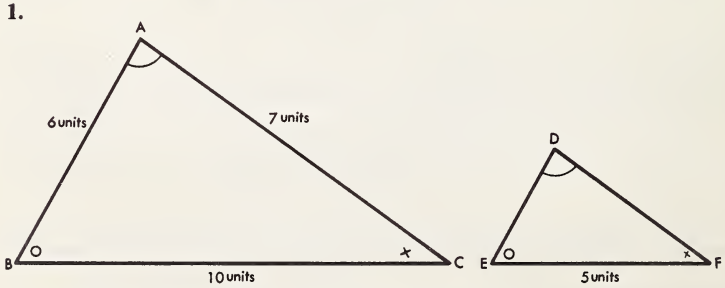
$y = 8$

Length of **DF** is 8 cm.

Exercise

A

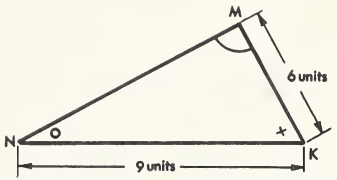
1.

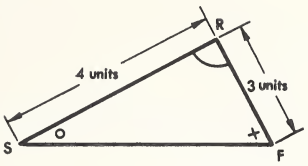


(*a*) **ABC** and **DEF** are two similar triangles with equal angles as indicated. Find the length of (*i*) **DE**, (*ii*) **DF**.

(*b*) If **EF** were 4 units in length, what would be the length of (*i*) **DE**, (*ii*) **DF**?

2.

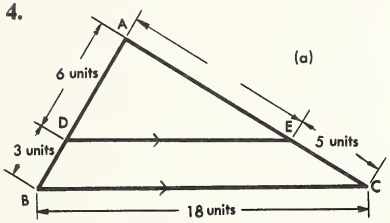


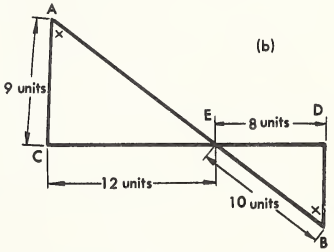


MNK and **RSF** are two similar triangles with angles equal as indicated. Find the length of (*i*) **MN**, (*ii*) **SF**.

3. A vertical stake with its top 3 feet above the ground casts a shadow 2 feet long.
   (*a*) At the same time, what will be the length of a shadow of a vertical pole 24 feet high?
   (*b*) What is the height of a tree which, at the same time, casts a shadow 30 feet long?

4.

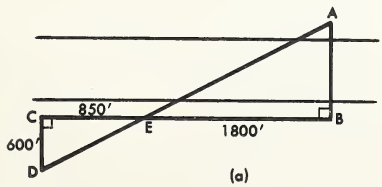


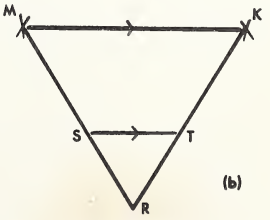


(*a*) Find the lengths of (*i*) **DE**, (*ii*) **AE**, (*iii*) **AC**.
(*b*) Find the lengths of (*i*) **AE**, (*ii*) **DB**, (*iii*) **AB**.

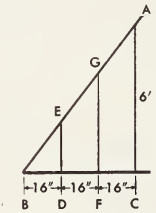
B

5. Use the measurements indicated on the diagram (*a*) and calculate **AB**, the distance across a river.





6. Find **MK** the length of a marsh as indicated on the diagram (*b*) given that **MR** is 5,000 feet, **RS** is 800 feet and **ST** which is parallel to **MK** is 1,250 feet.
7. A 36-foot ladder stands against a vertical wall with the base of the ladder 6 feet from the wall. How far from the wall is a rung which is 8 feet up the ladder?
8. A straight road has a gradient of 1 foot in 200 feet. How many feet, vertically, does a truck rise in travelling along this road between two points a horizontal distance of $\frac{1}{2}$ mile apart?
9. In a right-angled triangle **ABC**, $\angle \mathbf{B} = 90°$, **AB** $= 6''$ and **BC** $= 8''$. **BD** is drawn perpendicular to **AC** at **D**.
   (*a*) Show that each of the three triangles in this figure are similar to the other two.
   (*b*) Calculate the length of (*i*) **AC**, (*ii*) **AD**, (*iii*) **DC**, (*iv*) **BD**.
10. The diagram below represents the rafter lengths on a hip roof.



Find the length of each jack rafter, indicated by **ED** and **GF**.

Rectilinear Figures

A **rectilinear figure** is a plane figure enclosed by straight lines.

A **triangle** is a plane figure enclosed by three straight lines.

A **quadrilateral** is a plane figure enclosed by four straight lines.

A **trapezoid** is a quadrilateral having one pair of opposite sides parallel.

A **parallelogram** is a quadrilateral having both pairs of opposite sides parallel.

A **rhombus** is a parallelogram with all sides equal.

A **rectangle** is a parallelogram having one angle a right angle.

A **square** is a rectangle having all sides equal.

A **pentagon** is a plane figure enclosed by five straight lines.

A **hexagon** is a plane figure enclosed by six straight lines.

An **octagon** is a plane figure enclosed by eight straight lines.

Rectilinear figures of more than four sides are called **polygons**.

A **regular polygon** has all sides the same length and all angles equal.

Exercise

A

1. What is the sum of the three angles of any triangle?
2. If two of the angles of a triangle are 36° and 84°, what is the size of the third angle?
3. In triangle **ABC**, $\angle \mathbf{A} = \angle \mathbf{B}$ and $\angle \mathbf{C}$ is 70°. What is the size of the other angles?
4. In triangle **ABC**, $\angle \mathbf{B}$ is 80° and **AB** is produced to **S**. What is the size of the exterior $\angle \mathbf{SBC}$?

B

5. Construct any three quadrilaterals of different shapes. Measure each of the four angles in each quadrilateral and record the size of each angle on the diagram. Find the sum of the four angles of each quadrilateral.
6. Divide a quadrilateral into triangles by means of a diagonal and show that the sum of the angles of a quadrilateral is 360°.
7. If three angles of a quadrilateral are 36° 28′, 154° 45′ and 98° 24′, calculate the fourth angle.
8. If the three angles of a quadrilateral are 110° 8′ 50″, 24° 55′ 55″ and 87° 27′ 34″, calculate the fourth angle.
9. If two angles of a quadrilateral are 75° 47′ and 155° 37′ and the other two angles are equal, calculate the size of each of the equal angles.
10. Draw at least three pentagons. Measure, record and find the sum of the interior angles of each.
11. Divide a pentagon into triangles by means of diagonals drawn from one vertex. Calculate the sum of the angles of the pentagon.
12. Construct any pentagon **ABCDE**. Choose a point **O** inside the pentagon and join it to each of the five vertices.
    (*a*) How many triangles are formed?
    (*b*) By how much does the sum of the angles of the five triangles exceed the sum of the angles of the original pentagon?
    (*c*) State your observations regarding the sum of the angles of any pentagon.
13. If four angles of a pentagon are 45° 37′, 112° 3′, 145° 23′ and 140° 54′, calculate the remaining angle.
14. If three angles of a pentagon are right angles and the other two angles are equal, calculate the size of each of the equal angles.
15. The gable end of a garage is pentagonal in shape with each of the base angles 90°. If the angle at the top is to be 120°, what angle will the rafters make with the sides of the garage?

16. Calculate the sum of the interior angles of a hexagon by
(*a*) drawing diagonals from one vertex to divide it into triangles;
(*b*) joining each vertex to a point inside the hexagon.
State your conclusion regarding the sum of the angles of a hexagon.

17. Calculate the sum of the angles of an octagon.

18. Calculate the sum of the angles of a polygon of
(*a*) 7 sides, (*b*) 9 sides, (*c*) 10 sides,
(*d*) 12 sides, (*e*) 15 sides, (*f*) 16 sides.

19. Calculate, in terms of n, the sum of the angles of a polygon of n sides.

20. Draw a quadrilateral and produce each of the four sides in one direction only to form the four exterior angles. Find the sum of the exterior angles of a quadrilateral.

21. Find the sum of the exterior angles taken in order round
(*a*) a pentagon, (*b*) a hexagon, (*c*) an octagon.
State a general conclusion.

22. Calculate the size of each exterior angle of a regular polygon of
(*a*) four sides, (*b*) five sides, (*c*) six sides,
(*d*) eight sides, (*e*) ten sides, (*f*) sixteen sides.

23. Calculate the size of each angle of
(*a*) a regular pentagon, (*b*) a regular hexagon,
(*c*) a regular octagon, (*d*) a regular polygon of 20 sides.

24. Find an expression for the number of degrees in each angle of a regular polygon of n sides.

25. Copy and complete a table of the measures of regular polygons.

Kind of polygon	*Sum of the interior angles in degrees*	*Size of each interior angle in degrees*	*Size of each exterior angle in degrees*
Pentagon			
Hexagon			
Octagon			
Decagon			
Polygon of n sides ..			

C

26. If each angle of a regular polygon contains 150°, calculate the number of sides in it.

27. Find the number of sides in a regular polygon if each angle contains
(*a*) 144°, (*b*) 156°, (*c*) 157° 30′.

Properties of Polygons

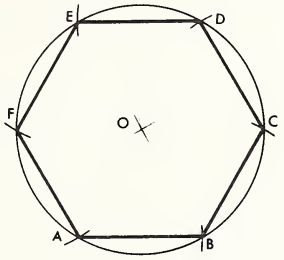
1. The sum of the interior angles of a polygon of n sides is $180(n - 2)$ degrees.

2. The sum of the exterior angles of any polygon formed by producing the sides, in order, is 360 degrees.

To Construct a Regular Hexagon Given One Side

Let one side of a hexagon be $\frac{3}{4}''$.

It is required to construct a regular hexagon with each side $\frac{3}{4}''$.



Method: With centre **O** and radius $\frac{3}{4}''$ construct a circle.

Choose a point **A** on the circumference and starting with centre **A** and radius $\frac{3}{4}''$ step off the arcs round the circle to get points **B**, **C**, **D**, **E** and **F**.

Join **AB**, **BC**, **CD**, **DE**, **EF**, **FA**.

ABCDEF is a regular hexagon with each side $\frac{3}{4}''$.

Exercise

B

1. Construct a regular hexagon with each side $2''$. Describe your method.

(*a*) Construct a perpendicular from the centre of the hexagon to one side. Measure and record its length.

(*b*) Check the accuracy of your measurement by calculating the length of the perpendicular, correct to one place of decimals.

(*c*) Find the distance across the flats of this hexagon.

2. Construct a regular hexagon with each side $1\frac{1}{2}''$.

(*a*) Calculate to two significant digits the distance from the centre to one of the sides.

(*b*) Calculate the area of the hexagon.

3. Draw any regular hexagon **ABCDEF**. Join **EA**.

(*a*) Calculate the size of $\angle$**EAF** and $\angle$**EAB**.

(*b*) Join **AC**, **CE** and **EA**. What type of triangle is formed?

4. Inscribe an equilateral triangle in a circle of diameter $2\frac{1}{2}''$. Measure and record the length of each side of the triangle.
5. Construct a regular hexagon if the distance across the flats is 2". Describe your method. Measure and record the length of each side.
6. Calculate the area of a regular hexagon the length of each side of which is as follows:
   (*a*) 20 inches, (*b*) 6 inches, (*c*) 56 centimetres.

C

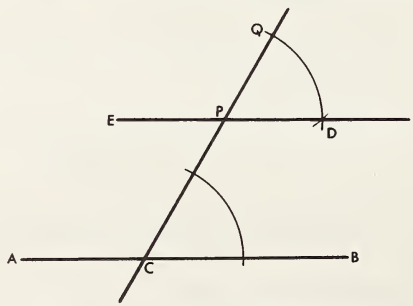
7. Construct a regular pentagon with each side $1\frac{1}{4}''$. Use your protractor.
8. The octagonal base for a lamp is cut from a square 12" to a side. Using a $\frac{1}{4}$ scale make a diagram of the square. To lay out the octagon, find the centre of the square by drawing the diagonals. With one vertex of the square as centre and radius equal to $\frac{1}{2}$ of the diagonal, draw arcs to cut the two adjacent sides. Repeat using as centre each of the other three vertices. Join the intersection points to form the octagon.
9. Lay out a regular octagon from a square 4" to a side.
10. (*a*) Inscribe a square in a 3" circle.
    (*b*) Inscribe a regular octagon in a $3\frac{1}{4}''$ circle.
    In each case, measure and record the length of each side of the regular figure.
11. Inscribe a regular pentagon in a 3" circle.

To Draw a Line Through a Point Parallel to a Given Line

(Alternative Method)

Let **AB** be a given line and **P** a given point.

It is required to draw a line through **P** parallel to **AB**.



Method: Through **P** draw any line to cut **AB** at **C**. At **P** make angle **QPD** equal to angle **PCB**. Produce **DP** to **E**.

ED is parallel to **AB**.

Exercise

B

1. Draw a horizontal line and mark a point above it. Through this point draw a line parallel to the given line. Describe your method.
2. Through a given point draw a line parallel to a given vertical line. Describe your method. Use method on page 35.
3. Draw a line **AB**. Construct a line **CD** parallel to **AB** and $\frac{3}{4}''$ above it.
4. Draw a line **MN**. Construct a line parallel to **MN** and $1\frac{1}{4}''$ below it.
5. In the trapezoid **ABCD**, **AD** is parallel to **BC**, $\angle \mathbf{B} = 50^\circ\ 30'$ and $\angle \mathbf{C} = 65^\circ\ 20'$. Calculate $\angle \mathbf{BAD}$ and $\angle \mathbf{ADC}$.
6. In any triangle **ABC**, a line **XAY** is drawn through **A** parallel to **BC**.
   (*a*) Which angles in the diagram are equal?
   (*b*) What is the sum of the angles **BAX**, **BAC** and **CAY**?
7. **AB** and **CD** are two straight lines which if produced would meet at 60°. **AB** and **CD** are to be joined by an arc of a circle with a $\frac{3}{4}''$ radius. Determine the position of the centre of this circle. Complete the construction.
8. The position of two lines **AB** and **CD** is such that, if they were produced, they would meet at right angles. They are to be joined by the arc of a circle with a radius of $\frac{1}{2}''$. Determine the centre of this arc and complete the construction.
9. Draw any triangle **ABC**. Bisect **AB** at **D**. Through **D** draw a line parallel to **BC** to meet **AC** at **E**. Measure and compare the lengths of **AE** and **EC**.
10. Draw any triangle **DEF**. Divide the base **EF** into four equal parts **EG**, **GH**, **HK** and **KF** respectively. Through **G**, **H** and **K** respectively, draw lines parallel to **DF** to cut **ED** at **M**, **N** and **R**. Measure and compare the lengths of **EM**, **MN**, **NR** and **RD**.

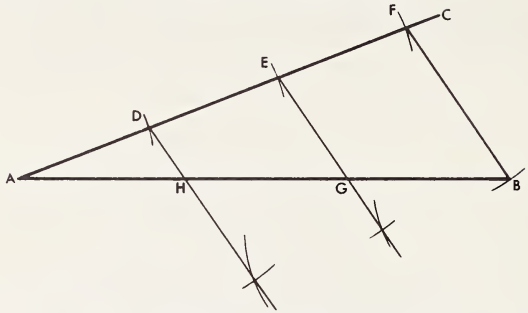
C

11. **AB** is parallel to **CD** and **EF** is a transversal cutting **AB** at **G** and **CD** at **H**. If $\angle \mathbf{BGH}$ and $\angle \mathbf{GHD}$ are bisected, determine the size of the angle between the bisectors when they are produced to meet.
12. Draw any two lines which, if produced, would intersect. Show how to connect them with an arc of a circle of radius 1″.
13. Construct an equilateral triangle $3\frac{1}{2}''$ to a side. Show how to "round the corners" with arcs each of radius $\frac{1}{2}''$.

To Divide a Line into Any Number of Equal Parts

Let **AB** be a given line.

It is required to divide **AB** into three equal parts.



Method: From **A** draw any line **AC**. Starting with centre **A** and any radius, cut off three equal parts **AD**, **DE** and **EF** along **AC**. Join **FB**.

Through **E**, draw **EG** parallel to **FB** to intersect **AB** at **G**.

Similarly draw through **D** a line parallel to **FB** to cut **AB** at **H**.

AH, **HG** and **GB** are the three equal parts of **AB**.

Exercise

B

1. Divide a line 5″ long into three equal parts. Describe your method.
2. Divide a line 6″ long into 5 equal parts. Describe your method.
3. Draw a vertical line **CD** 3″ long. Divide **CD** into 5 equal parts at **E**, **F**, **G** and **H** respectively. What is the ratio of
   (*a*) **CE** to **ED**; (*b*) **CF** to **FD**; (*c*) **CE** to **CD**; (*d*) **CF** to **CD**?
4. Divide a line 4″ long in the ratio of 2 : 3.
5. Divide a line 4″ long in the ratio of 1 : 2.
6. Divide a line $3\frac{1}{2}$″ long in the ratio of 1 : 2 : 3.
7. The perimeter of a triangle is 6″. The sides are in the ratio of 2 : 3 : 3. Construct the triangle.
8. The perimeter of a rectangle is 10″ and the ratio of the lengths of the sides is 2 : 1. Construct the rectangle.

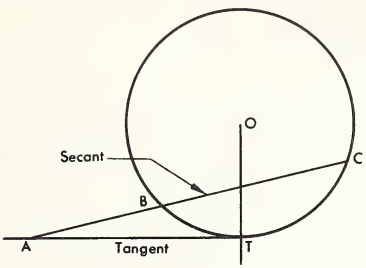
Definitions

A **secant** of a circle is a line drawn from a point outside a circle to cut the circle at two points.

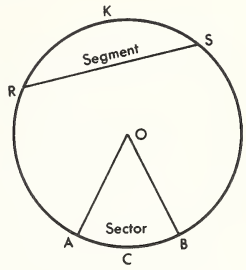
A **tangent** of a circle is a line drawn from a point outside a circle such that it touches the circle at one point only.

A **sector** is a part of a circle bounded by two radii and a connecting arc.

A **segment** of a circle is a part cut off by a chord. The smaller part is the **minor segment**; the larger part is the **major segment.**



ABC is a secant of the circle.
AT is a tangent of the circle touching the circle at **T**, the **point of contact.**



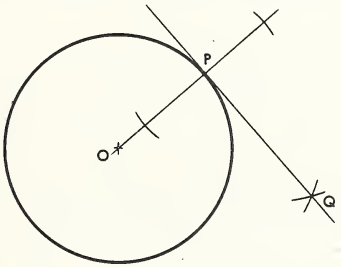
AOB is a sector of the circle.
The arc **ACB subtends** the $\angle$**AOB** at the centre.
RKS is a segment of the circle.

It can be shown that the tangent of a circle is perpendicular to the radius drawn to the point of contact.

To Draw a Tangent to a Circle at a Point on the Circumference

Let **P** be a point on the circumference of a circle with centre **O.**

It is required to draw through **P** a tangent to the circle.



Method: Join **OP.** Construct **PQ** perpendicular to **OP.**

PQ is a tangent to the circle.

Exercise

B

1. Draw a circle of radius $1\frac{1}{2}$". Mark a point **A** on the circumference. Draw a tangent to this circle at point **A.** Describe the method.

2. Mark any three points **A**, **B** and **C** not in a straight line.
   (*a*) Draw a circle to pass through **A**, **B** and **C**.
   (*b*) Draw a tangent to this circle at **B**.
3. Draw a triangle **ABC** having **AB** = 3″, $\angle$**A** = 75° and $\angle$**C** = 45°.
   (*a*) Circumscribe a circle about this triangle.
   (*b*) At **B** draw **BD** a tangent to the circle.
   (*c*) Measure the angle **CBD**.
4. Draw a line **AB** and mark on it a point **C**. Through **C** draw a circle of radius 2″ such that **AB** will be tangent to this circle.
5. Draw a straight line **AB**. Mark **C** a point on **AB** and **D** a point not on **AB**. Draw a circle to pass through **C** and **D** and have **AB** as a tangent.
6. Construct two concentric circles of radii $2\frac{1}{2}$″ and $1\frac{1}{2}$″. At any point on the circumference of the inner circle construct a tangent and produce it to meet the outer circle at **A** and **B**. Calculate the length of **AB**.
7. Draw a circle of radius 2″. Construct **AB** a chord 3″ long in this circle. On the arc of the major segment, mark any three points **C**, **D** and **E**. Construct and measure the angles **ACB**, **ADB** and **AEB**.
8. Construct a diagram to illustrate that all the angles in the same segment of a circle are equal.
9. Draw any circle with centre **C**. Mark off an arc **AB**. Draw the angle **BCA** and compare it with at least three angles subtended by the arc **AB** at the circumference. State a possible conclusion.

CHAPTER 20

Mensuration of Triangles

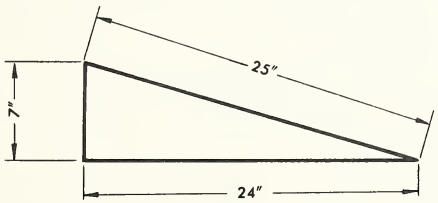
To Determine the Area of a Triangle when the Lengths of the Three Sides are Known

The Right-angled Triangle

When the lengths of the three sides satisfy the right-angled triangle relationship, the area may be found by applying the formula

$$A = \tfrac{1}{2}bh.$$

Example: Calculate the area of a triangle if the lengths of the sides are 7″, 24″ and 25″ respectively.



Solution: Check to determine if the triangle is right-angled.

The sum of the areas of the squares on the two shorter sides in sq. in. is $7^2 + 24^2 = 625$.

The square on the longest side in sq. in. is $25^2 = 625$.

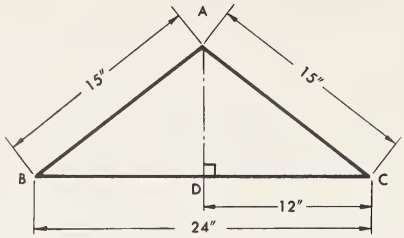
Therefore, the triangle is right-angled with the length of the base 24″ and altitude 7″.

The area in sq. in. is $\frac{1}{2} \times 24 \times 7 = 84$.

The Altitude and the Area of an Isosceles Triangle

Since the perpendicular drawn from the vertex of an isosceles triangle to the base bisects the base, the altitude can be calculated when the lengths of the sides are known.

Example: Find the area of the isosceles triangle **ABC** in which the length of the base **BC** is 24″ and each of the equal sides is 15″.



Solution:

Draw **AD** perpendicular to **BC**. The length of **DC** is 12″.

In the right-angled triangle **ADC**,

the length of **AD** in inches is $\sqrt{15^2 - 12^2} = 9$.

Therefore, the area of the triangle **ABC** in sq. ft. is $\frac{1}{2} \times 24 \times 9 = 108$.

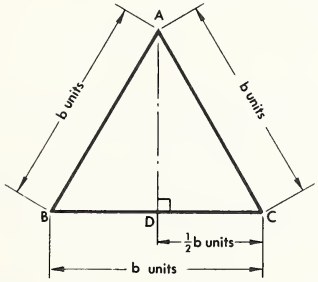
Exercise B

1. Show that triangles with lengths of sides as follows are right-angled and calculate the area of each:
   (*a*) 9′, 40′ and 41′; (*b*) 12″, 35″ and 37″;
   (*c*) 16″, 63″ and 65″; (*d*) 24 cm., 45 cm. and 51 cm.
2. Calculate the area of each of the following isosceles triangles the lengths of the sides of which are:
   (*a*) 13″, 13″ and 10″; (*b*) 20″, 20″ and 32″;
   (*c*) 17″, 17″ and 16″; (*d*) 22″, 61″ and 61″.
3. Calculate the area of the isosceles triangles the lengths of the sides of which are as follows:
   (*a*) 10″, 10″ and 6″; (*b*) 6·0′, 6·0′ and 4·8′;
   (*c*) 100′, 100′ and 180′.
4. The roof of a church tower is formed by four equal isosceles triangles each with a base of 12′ and with equal sides of 24′. Find the area of the roof.
5. On a building with a gable roof, the rafters without overhang are 13 feet long. If the span is 20 feet, find
   (*a*) the height of the ridge board above the plate;
   (*b*) the area of the triangle between the ridge and the plate level.
6. If a chord 8″ long is drawn in a 10″ circle, how far is this chord from the centre of the circle?
7. A chord 22″ long is drawn in a circle of radius 61″. Find the distance of this chord from the centre.

8. What is the length of a chord 5″ from the centre of a 26″ circle?
9. What is the length of a chord 15″ from the centre of a 34″ circle? What is the area of the triangle formed by joining the ends of the chord to the centre?
10. Calculate the areas of the equilateral triangles, the lengths of each side of which are as follows:
(*a*) 2″, (*b*) 10″.

The Altitude and the Area of an Equilateral Triangle

The length of each side of an equilateral triangle is b linear units.
Find the formula for
(*a*) its altitude, (*b*) its area.



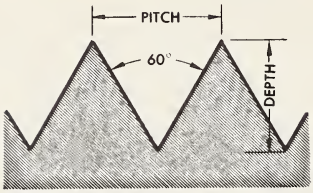
Let **ABC** be an equilateral triangle.
Draw **AD** perpendicular to **BC**.

(*a*) The length of **DC** in linear units is $\frac{1}{2}b$.
Let the length of **AD** in linear units be h.
From the right-angled triangle **ADC**
the length of **AD** in linear units is $\sqrt{b^2 - (0{\cdot}5b)^2}$
$= \sqrt{0{\cdot}75b^2}$
$= 0{\cdot}866b$.
Therefore, $h = 0{\cdot}866b$.

(*b*) Let the area of the equilateral triangle **ABC** in sq. units be A.
But the area of the triangle **ABC** in sq. units is $\frac{1}{2}b \times 0{\cdot}866b$
$= 0{\cdot}433b^2$.
Therefore, $A = 0{\cdot}433b^2$.

Example 1: If the sharp V thread shown in the diagram has 8 threads per inch, calculate the depth of the thread.

Solution: The cross-section of a sharp V thread is an equilateral triangle with the pitch of the thread as base.



The pitch of the thread in inches is $\frac{1}{8}$.

The depth of the thread in inches is $0{\cdot}866 \times \frac{1}{8} = 0{\cdot}108$.

Example 2: Calculate the area of an equilateral triangle each side of which is 6″.

Solution: The area of the equilateral triangle in sq. in. is $0{\cdot}433 \times 6^2 = 15{\cdot}6$.

Exercise

B

1. Calculate the altitudes of each of the equilateral triangles of which the lengths of the sides are as follows:
   (*a*) 20″, (*b*) 4·5″, (*c*) $3\frac{1}{8}$″.
2. Calculate the depth of each of the sharp V threads which have the following number of threads per inch:
   (*a*) 10, (*b*) 24, (*c*) 16, (*d*) 6.
3. Find the area of each of the following equilateral triangles:
   (*a*) each side 6″; (*b*) each side 8 cm.;
   (*c*) each side 2·5″; (*d*) each side $4\frac{1}{8}$″.
4. The depth of the commonly used American National Thread is $\frac{3}{4}$ that of a sharp V thread of the same pitch. Find the depth of each of the American National threads described as follows:
   (*a*) $\frac{1}{4}$″ — 20 — N.C.; (*b*) $\frac{1}{2}$″ — 13 — N.C.;
   (*c*) 1″ — 8 — N.C.; (*d*) 2″ — $4\frac{1}{2}$ — N.C.

Note: 1″—8—N.C. describes a National Coarse thread for a 1″ diameter bolt with 8 threads per inch.

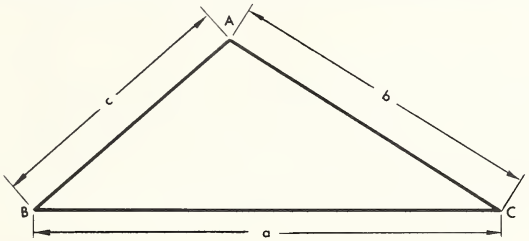
5. Calculate the length of each side of an equilateral triangle the height of which is as follows:
   (*a*) 3·464 in.; (*b*) 8·66 in.; (*c*) 1·00 in.
6. Calculate the distance across the flats of a regular hexagon each side of which is
   (*a*) 2″, (*b*) 0·5″, (*c*) 0·75″, (*d*) 1·25″.
7. Calculate the side of a regular hexagon if the distance across the flats is
   (*a*) 10·392″, (*b*) 6·00″, (*c*) 1·50″.

8. Calculate the area of a regular hexagon if the lengths of its sides are as follows:

   (*a*) 1″, (*b*) 3″, (*c*) 10″, (*d*) 5·5″.

9. Calculate the area of a regular hexagon if the distance across the flats is 17·32″.

The Area of Any Triangle for which the Lengths of the Three Sides are Known



For any triangle **ABC** let the measures of the lengths of the sides opposite the vertices **A**, **B** and **C** be *a*, *b* and *c* respectively.

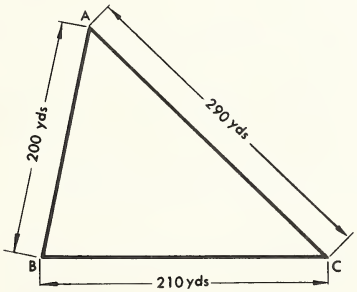
Let the measure of the semi-perimeter be *s*.

Let the measure of the area in square units be *A*.

Then, the measure of the area may be found by applying the formula

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Example: A small island is approximately triangular in shape. If its three sides measure 210 yd., 200 yd. and 290 yd. respectively, find its area.



Solution:

$a = 210$ $\quad s - a = 350 - 210 = 140$

$b = 290$ $\quad s - b = 350 - 290 = 60$

$c = 200$ $\quad s - c = 350 - 200 = 150$

$2s = \overline{700}$ $\quad$ *Check:* $\overline{3s - (a + b + c)} = \overline{350}$

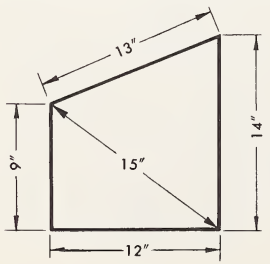
$s = 350$ $\quad s = 350$

The formula for the area is $A = \sqrt{s(s-a)(s-b)(s-c)}$

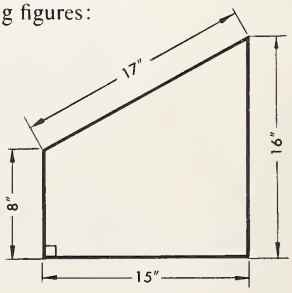
The area of the triangle in sq. yd. is $\sqrt{350 \times 140 \times 60 \times 150}$
$= 21{,}000.$

Exercise
B

1. Find the area of a triangle the lengths of the sides of which are 13″, 14″ and 15″.
2. An isosceles triangle has sides of 40″, 40″ and 64″. Find
   (*a*) its area by using the *s*-formula;
   (*b*) its area by first finding its altitude to the 64″ side.
3. The sides of a triangle are 5″, 12″ and 13″. Find its area
   (*a*) by using the *s*-formula;
   (*b*) by first showing that it is right-angled.
4. Find the area of a triangle if the lengths of its sides are 20″, 48″ and 52″.
5. (*a*) Calculate the area of a triangle the lengths of the sides of which are 25″, 60″ and 65″.
   (*b*) Calculate the altitude drawn to the 65″ side.
6. The lengths of the sides of a triangle are 29″, 36″ and 25″. Calculate,
   (*a*) the area of the triangle;
   (*b*) the length of each of the three altitudes.
7. Calculate the areas of the triangles the lengths of the sides of which are as follows:
   (*a*) 3″, 5″ and 6″; (*b*) 4′, 20′ and 18′;
   (*c*) 10 cm., 6 cm. and 8 cm.; (*d*) 90 m., 80 m. and 26 m.
8. In a certain survey, a triangle has sides of 25 chains, 17 chains and 12 chains. Calculate the area of this triangle in acres.
9. The lengths of the sides of a triangle are 13″, 14″ and 15″ respectively. How is the area of this triangle altered if the lengths of the sides are
   (*a*) doubled, (*b*) trebled, (*c*) halved?
10. Find the area of each of the following figures:



(a)



(b)

11. The areas of similar triangles are proportional to the squares on corresponding sides.

(*a*) If a triangle with an area of 490 sq. ft. has a base of 35 ft., what will be the area of a similar triangle with a base 30′ long?

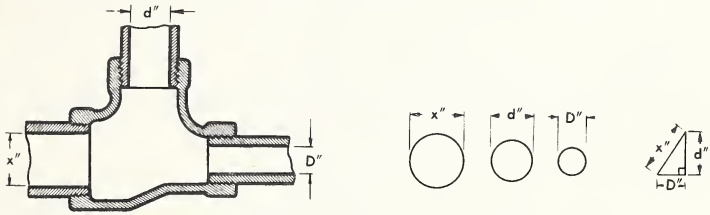
(*b*) If the base of a triangle with an area of 375 sq. ft. is 40′ in length, what is the length of the base of a similar triangle which has an area of 240 sq. ft.?

Applications of the Right-angled Triangle Law

Capacity of Pipes

For most practical purposes it may be assumed that the carrying capacity of a pipe is proportional to the area of its cross-section.

To Determine the Size of a Pipe which will have the Same Capacity as that of Two Given Pipes



Let the diameters of the two pipes in linear units be d and D.

Let the diameter of the third pipe in linear units be x.

The sum of the areas of the cross-sections of the first two pipes in square units is $\dfrac{\pi d^2}{4} + \dfrac{\pi D^2}{4}$.

The area of the cross-section of the new pipe in sq. units is $\dfrac{\pi x^2}{4}$.

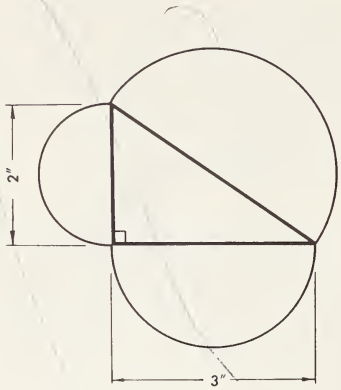
Therefore, $$\frac{\pi x^2}{4} = \frac{\pi d^2}{4} + \frac{\pi D^2}{4}.$$

$$x^2 = d^2 + D^2.$$

This relationship is the same as that for the right-angled triangle in which x is the measure of the hypotenuse and d and D are the measures of the two sides respectively.

In the calculations that follow the nominal size of a pipe will be considered as its internal diameter. The nominal sizes of standard pipes are as follows: *The diameters are in* $\frac{1}{8}$″ *intervals up to 1″ except* $\frac{7}{8}$″; $\frac{1}{2}$″ *intervals from 1″ to 5″ and in inch intervals above 5″.*

Example: Calculate the diameter of a pipe which will carry as much water as a 2″ and a 3″ pipe combined.



Solution:

The diameter of the pipe in inches is $\sqrt{2^2 + 3^2} = \sqrt{13}$
$= 3{\cdot}6.$

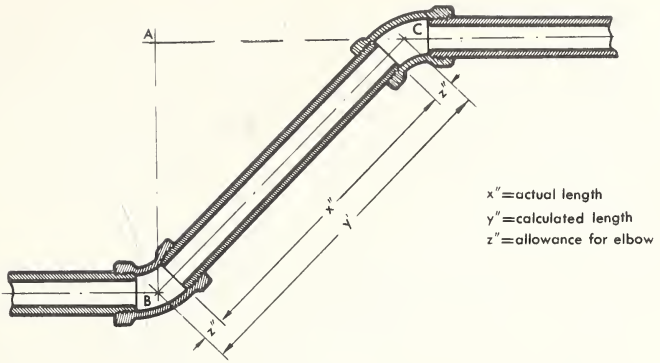
In this case a 4″ pipe would be used.

Exercise

B

1. Find the diameter of a pipe required to carry the flow from a 3″ and a $3\frac{1}{2}$″ pipe. What standard size pipe would be used?
2. Calculate, to one place of decimals, the size of a pipe which will carry as much water as each of the following:
   (*a*) two $\frac{1}{2}$″ pipes; (*b*) two 1″ pipes;
   (*c*) a $\frac{1}{2}$″ pipe and a 1″ pipe; (*d*) a 4″ pipe and a 6″ pipe.
3. Calculate to one place of decimals, the size of a pipe which will carry as much water as each of the following:
   (*a*) three 1″ pipes; (*b*) a 1″ pipe, a 2″ pipe and a 3″ pipe;
   (*c*) three 2″ pipes; (*d*) two 3″ and a 5″ pipe.
4. A steam heating system uses a 3″, a 4″ and a 5″ circular pipe. What size pipe will carry as much steam as these combined?
5. If a 5″ ventilating pipe were already supplying a 3″ pipe, what other size of pipe could be taken off the 5″ line?
6. In a waterworks system, how many $\frac{1}{2}$″ pipes could be supplied by a 6″ pipe if they were all to operate at full capacity at the same time?

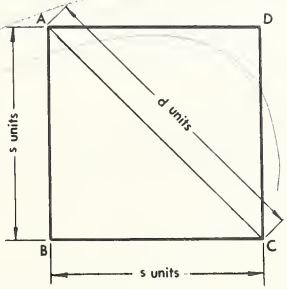
7.



In the diagram above, the change in direction of a pipe line by means of a 45° offset is shown. Find the calculated length of the offset pipe if the measured length of **AB** is as follows:

(*a*) 12″, (*b*) 18″, (*c*) 13½″, (*d*) 16¼″.

The Diagonal of a Square



Let the length of the square in linear units be s.

Let the length of the diagonal of the square in the same units be d.

From the right-angled triangle **ABC**, $d^2 = s^2 + s^2$.

$$d^2 = 2s^2.$$

Take the square root of each side, $d = s\sqrt{2}$.

$$\boldsymbol{d = 1{\cdot}414s}.$$

Explain how this formula may be solved for s to obtain

$$\boldsymbol{s = 0{\cdot}707d}.$$

Example: Determine the distance from home base to second base on a softball diamond which is a square 45′ to each side.

Solution: The relationship is expressed by the formula $d = 1{\cdot}414s$.

The distance from home to second base in feet is $1{\cdot}414 \times 45 = 63{\cdot}6$.

Exercise
B

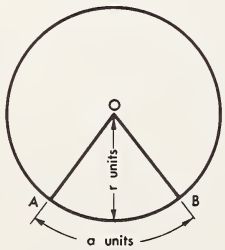
1. Find the length of the diagonal of a square if the length of each side is
   (*a*) 5″, (*b*) 10″, (*c*) 1″, (*d*) 2″, (*e*) 3″, (*f*) 25″.
2. Square ends are to be milled on round stock. Calculate, to the nearest hundredth of an inch, the diameter of the stock necessary if the length of the square is to be:
   (*a*) $\frac{1}{2}$″, (*b*) $\frac{1}{4}$″, (*c*) 0·625″, (*d*) 1·25″.
3. Find the depth of cut necessary to mill each square in problem 2.
4. Calculate the length of the side of a square if the length of the diagonal is:
   (*a*) 4″, (*b*) 15″, (*c*) 20″, (*d*) 1″, (*e*) 2″, (*f*) *x*″.
5. (*a*) What is the size, to the nearest hundredth of an inch, of the largest square which can be cut from round stock if the diameter of the stock is as follows:
   (i) $\frac{1}{2}$″, (ii) $\frac{3}{4}$″, (iii) $1\frac{1}{2}$″, (iv) $3\frac{1}{2}$″?
   (*b*) For each case above, state the depth of cut necessary to mill the square end.
6. Square heads on bolts are 1″ across the flats. What size hole should be bored to countersink these heads if $\frac{1}{8}$″ clearance between the corners of the head and the wood is to be left?

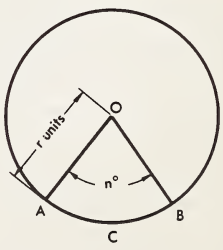
Sector of a Circle

For the purpose of calculating the area of a sector of a circle we may consider it as a triangle with a base equal to the arc and an altitude equal to the radius.

Let the area of the sector in square units be *A*.

Let the length of the arc in linear units be *a*.





Let the radius of the arc in linear units be *r*.

Then the measure of the area is given by the formula

$$A = \tfrac{1}{2}ar.$$

Example 1: Find the area of the sector of a circle of radius 5″ and arc 3″.
Solution: The area of the sector in sq. in. is $\frac{1}{2} \times 5 \times 3 = 7{\cdot}5$.

Again, if the angle of a sector is $n°$, it can be shown that

(i) the arc of the sector is $\frac{n}{360}$ of the circumference of the circle;

(ii) the area of the sector is $\frac{n}{360}$ of the area of the circle.

Example 2: A 50° sector is cut from a 12″ circle. Find
(*a*) the length of the arc, (*b*) the area of the sector.
Solution: The circumference of the circle in inches is

$$3{\cdot}14 \times 12.$$

The length of the arc of the sector in inches is

$$\frac{50}{360} \times 3{\cdot}14 \times 12 = 5{\cdot}2.$$

The area of the sector in sq. in. is $\frac{50}{360} \times 3{\cdot}14 \times 6^2 = 15{\cdot}7$.

Exercise

B

1. Calculate the area of the sectors of circles with dimensions as follows:
(*a*) length of arc 3″, radius of arc 8″;
(*b*) length of arc 5·4 cm., radius of arc 6·8 cm.;
(*c*) length of arc $5\frac{3}{4}$″, radius of arc $8\frac{7}{8}$″;
(*d*) length of arc 5 ft., radius of arc 8 yd.

2. Find the ratio of the areas of two sectors in the same 10″ circle if the arc of the first sector is $2\frac{1}{2}$″ and the arc of the second sector is $3\frac{3}{4}$″.

3. Find the length of the arc of a sector of a 10″ circle subtended by an angle of 60° at the centre.

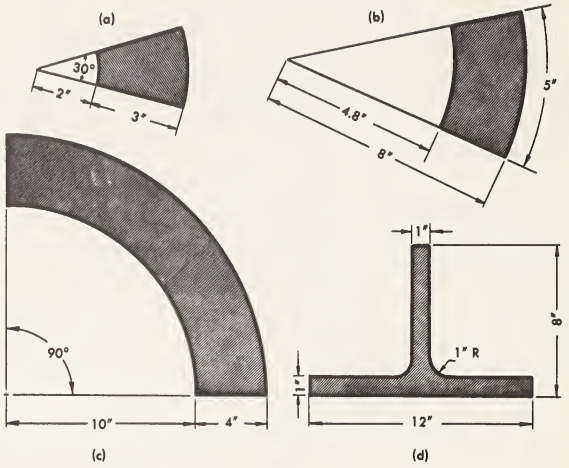
4. If a pendulum 24″ long swings through an angle of 10°, find the length of the arc traced out by the end of the pendulum.

5. Calculate the length of the arc of a quadrant of an 8″ circle.

6. Find the difference between the lengths of the chord and its corresponding arc when
(*a*) a regular hexagon is inscribed in a 6″ circle,
(*b*) a square is inscribed in a 4″ circle.

7. Calculate the area of a 150° sector in a 6″ circle.

8. Find the area of the shaded portion of each of the following diagrams:



9. Construct a sector of a circle with radius of 2″ and containing an angle of (*a*) $67\frac{1}{2}°$, (*b*) 75°, (*c*) 90°, (*d*) 135°, (*e*) 225°, (*f*) 330°.

C

10. Construct a segment of a circle $\frac{1}{2}$″ in height on a chord 3″ long.

(*a*) Measure and record the length of the radius.

(*b*) Check the accuracy of your construction by calculating the radius.

11. Construct a segment of a circle with:

(*a*) a height of 4·5″ on a chord 3″ long;

(*b*) a height of $1\frac{1}{8}$″ on a chord $3\frac{3}{4}$″ long;

(*c*) a height of 4″ on a chord 2″ long;

(*d*) a height of 1″ on a chord 4″ long;

(*e*) a height of 4″ on a chord 4″ long.

Measure the radius of the circle in each case. By calculation check the accuracy of each construction.

Review Exercise

1. Determine the area of the right-angled triangle **ABC** if the sides are 11″, 60″ and 61″.

2. Find the area of a triangle with sides of 39″, 42″ and 45″.

3. An isosceles triangle has equal sides of 48″ and a base of 16″. Determine its altitude and its area.

4. Find the height of an equilateral triangle if each side is 12 inches.

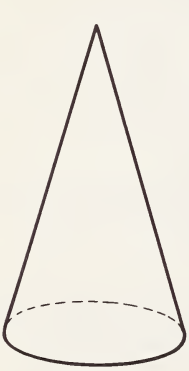
5. Find the area of an equilateral triangle each side of which is 8″.
6. Find the depth of a National Coarse thread with 12 threads per inch.
7. Determine the size of a pipe which will have the same carrying capacity as 8 half-inch pipes.
8. Calculate the diagonal of a square each side of which is 90 feet.
9. What is the largest square which can be cut from round stock $\frac{3}{4}$ inch in diameter?
10. Calculate the arc and area of a sector if its angle is 30° and it is cut from a circle 21″ in diameter. Use $\pi = \frac{22}{7}$.
11. If the arc of a sector is 8″ long and the radius is 7″, determine the area of the sector.
12. Find the length of the arc of a quadrant of a 10″ circle.

CHAPTER 21

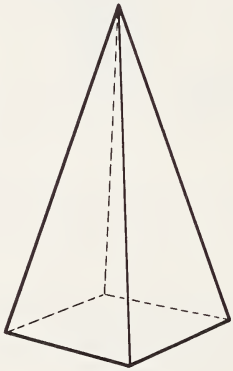
The Cone, Pyramid and Sphere

Definitions

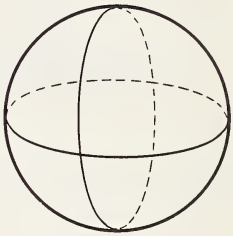
A **right circular cone** is a solid, the base of which is a circle and the sides of which taper to a vertex directly over the centre of the base.



CONE



PYRAMID



SPHERE

A **right pyramid** is a solid, the sides of which are isosceles triangles and the base of which is any regular polygon.

A **sphere** is the geometrical name of a round or ball-shaped solid. Every point on the surface of a sphere is equidistant from its centre.

The Area of the Curved Surface of a Cone

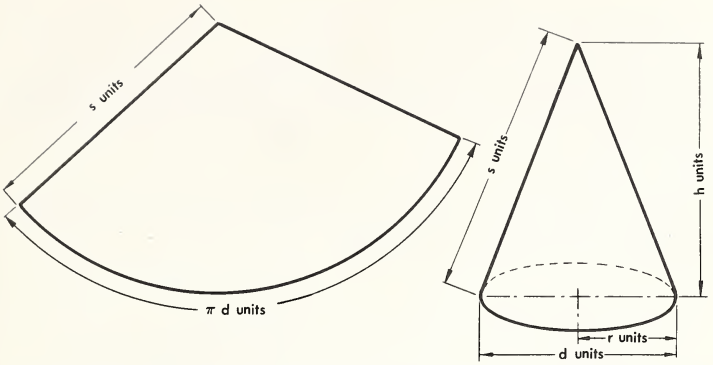
If a sector of a circle is rolled in such a way that the two boundary radii are brought together, a cone is formed. The arc of the sector becomes the circumference of the base of the cone and the radius of the sector becomes the slant height of the cone.

The measure of the altitude of the cone is h.

The measure of the radius of the base of the cone is r.

The measure of the diameter of the base of the cone is d.

The measure of the slant height of the cone is s.



Since the area of the curved surface of the cone is equal to the area of the corresponding sector, it can be shown that the relationship for finding the curved surface of a cone is expressed by the formula

$$A = \tfrac{1}{2}\pi ds,$$

or

$$A = \pi rs.$$

Example: If the slant height of a cone is 10″ and the diameter of the base is 12″, calculate the area of the curved surface.

Solution: The required relationship is $A = \frac{1}{2}\pi ds$.

The area of the curved surface in sq. in. is $\frac{1}{2} \times 3{\cdot}14 \times 12 \times 10 = 188{\cdot}4$.

Exercise B

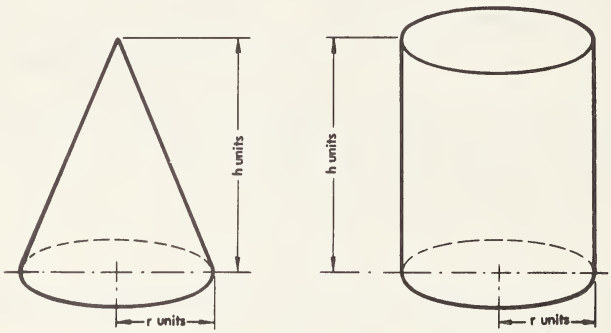
1. If a sector with an arc 15″ long is cut from an 8″ circle and rolled to form the curved surface of a cone, calculate
   (*a*) the area of the curved surface,
   (*b*) the diameter of the base of the cone.
2. Find the area of the curved surface of each cone of which the dimensions are as follows:
   (*a*) slant height 10″, diameter of base 3″;
   (*b*) slant height 8″, radius of base 4″;
   (*c*) slant height 15″, diameter of base 8″.
3. If the altitude of a cone is 24″ and the diameter of the base is 14″, calculate
   (*a*) the slant height, (*b*) the lateral area.
4. Calculate the area of the curved surface of a cone if the altitude is 40″ and the diameter of the base is 18″.
5. Calculate the entire area of the surface of a cone if its dimensions are as follows:

(*a*) slant height 8′, diameter of base 10′;
(*b*) altitude 12″ and diameter of the base 10″.

6. A cylindrical silo 16′ in diameter has a conical roof with a rise of 6′. Calculate the area of the entire outside surface of this silo if the over-all height is 36′.

Volume of a Cone

It can be shown that the volume of a cone is $\frac{1}{3}$ the volume of a cylinder with the same base and the same height.



A Cone and Its Corresponding Cylinder

Since the volume of a cylinder is given by the formula $V = \pi r^2 h$, the relationship for finding the volume of a cone is expressed by the formula

$$V = \tfrac{1}{3}\pi r^2 h.$$

Example: If the diameter of the base of a cone is 3″ and its altitude is 5″, calculate its volume.

Solution: The required relationship is $V = \frac{1}{3}\pi r^2 h$.

The radius of the base in inches is $\frac{3}{2} = 1{\cdot}5$.

The volume of the cone in cu. in. is $\frac{1}{3} \times 3{\cdot}14 \times 1{\cdot}5^2 \times 5 = 11{\cdot}8$.

Exercise

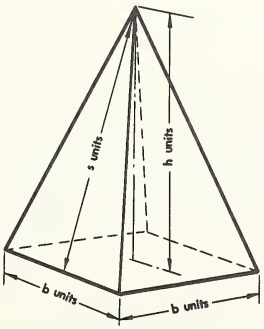
B

1. Calculate the volume of the cones with the following dimensions:
   (*a*) altitude 9 feet, diameter of the base 10 feet;
   (*b*) altitude 8 feet, diameter of the base 6 feet;
   (*c*) altitude 2 feet, radius of the base 9 feet.
2. A pile of sand dumped from elevated tracks assumed a conical shape. If the height of the pile is 10 feet and the diameter of the base is 24 feet, calculate the volume of the pile.

3. What will be the capacity, in pints, of an oil container in the shape of an inverted cone with a depth of 10″ and a diameter across the top of 4″?
4. A grain bin has the form of a cylinder with a conical base. If the over-all height is 20′, the diameter is 10′ and the depth of the conical part is 8′, find its capacity in bushels.
5. The slant height of a cone is 17 feet. If the diameter of its base is 16 feet, calculate (*a*) its altitude, (*b*) its volume.
6. Syrup from a 5-gallon drum is put into conical containers 4″ in diameter and 6″ in height. How many such containers can be filled from a full drum?
7. What is the ratio of the volume of a cone 6″ in diameter and 10″ high to that of a cylinder 12″ in diameter and 20″ high?
8. How many cubic feet of air are in a conical tent if the diameter of the base is 18 feet and the slant height is 15 feet?
9. A standard lathe centre has a carbide tip which is in the shape of a cone. How many cubic inches of carbide will be required if the diameter at the base is 0·580 inches and it is 0·500 inches high?
10. Iron ore is stacked in a conical shaped pile. How many cubic yards of ore are in a stack if the diameter of the base is 95 feet and the height is 55 feet?

The Lateral Area of a Right Pyramid

The lateral area of a pyramid is the sum of the areas of the isosceles triangles which make up its sides. When the base of the pyramid is a regular polygon all the triangular faces are identical.



A Square Pyramid

The measure of the slant height of the pyramid is *s*.

The measure of the altitude of the pyramid is *h*.

The measure of the side of the base of the pyramid is b.

If the base has n sides there will be n isosceles triangles and the relationship for the lateral area is expressed by the formula

$$A = \tfrac{1}{2}bsn.$$

Example: If the base of a pyramid is a square 6″ to a side and the slant height is 8″, calculate its lateral area.

Solution: The area of one triangular face in sq. in. is $\frac{1}{2} \times 6 \times 8$.

The lateral area of the pyramid in sq. in. is $\frac{1}{2} \times 6 \times 8 \times 4 = 96$.

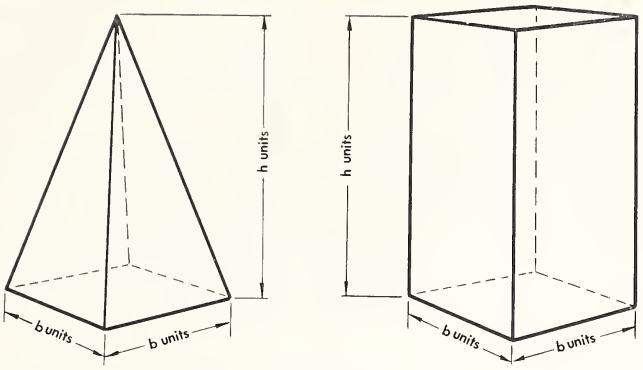
Exercise

B

1. Calculate the area of the roof of a tower which is in the form of a square pyramid, each side of which is 16 feet and the slant height of which is 12 feet.
2. If the altitude of a pyramid is $7\frac{1}{2}$ feet and it stands on a square base each side of which is 8 feet, calculate
   (*a*) the slant height, (*b*) the lateral area of the pyramid.
3. A brass paper-weight in the form of a square pyramid has a base 2″ on each side and a slant height of 3″. Find the area of the lateral surface.
4. Find the total surface of a square pyramid of base 9″ and of slant height 12″.
5. What is the area of the lateral surface of a hexagonal pyramid if each side of the base is 6″ and the slant height is 12″?
6. Find the slant height of a square pyramid of which the base is 18″ and the altitude is 40″.
7. The distance measured along one of the edges of a triangular face of a square pyramid is 41″. If the side of the base is 18″, calculate the slant height.
8. If the altitude of a square pyramid is 60″ and each side of the square base is 22″, calculate
   (*a*) the slant height, (*b*) the length of an edge, (*c*) the lateral area.
9. Find the area of the lateral surface of a rectangular pyramid if the measures of the sides of its base are 18″ and 10″ respectively and its altitude is 12″.
10. A pyramid is on a rectangular base 20″ by 14″. If the altitude is 24″, calculate the area of its lateral surface.

The Volume of a Pyramid

It can be shown that the volume of a pyramid is $\frac{1}{3}$ the volume of a prism with the same base and the same height.



A Pyramid and Its Corresponding Prism

Let the measure of the area of the base of the pyramid be A.

Since the volume of a prism is given by the formula $V = Ah$, the relationship for finding the volume of a pyramid is expressed by the formula

$$V = \tfrac{1}{3}Ah.$$

Example: Calculate the weight of a small stainless steel paper holder in the form of a square pyramid of which each side of the base is 2″ and the altitude is $2\frac{1}{2}$″. The weight of steel is 0·28 lb. per cu. in.

Solution: The relationship for the volume is $V = \frac{1}{3}Ah$.

The volume of the pyramid in cu. in. is $\frac{1}{3} \times 2^2 \times 2{\cdot}5$.

The weight of the holder in lb. is $\frac{1}{3} \times 4 \times 2{\cdot}5 \times 0{\cdot}28 = 0{\cdot}93$.

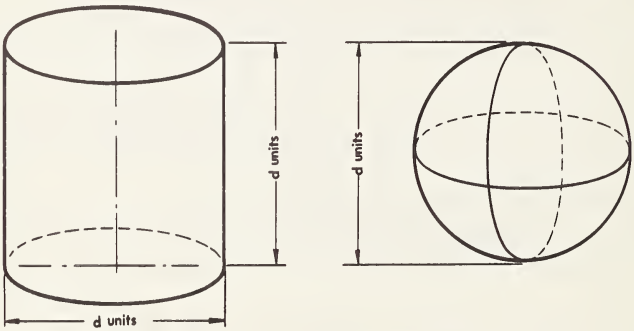
Exercise B

1. Find the volume of each of the following pyramids:
   (*a*) a square base 8′ to the side and an altitude of 6′;
   (*b*) a square base 6′ 4″ to the side and an altitude of 7′ 3″;
   (*c*) a square base 24′ to the side and a slant height of 37′.
2. A cement hopper is in the form of an inverted pyramid. Its rectangular top is 8′ by 6′ and the hopper is 10′ deep. How many bags of cement will it hold if each bag contains approximately 1 cubic foot?
3. Find the volume of a pyramid if it is on a rectangular base 3′ by 8′ and has an altitude of 15′.
4. Find the volume of a pyramid if it is on a triangular base with sides 13″, 14″ and 15″ and has an altitude of 20″.
5. Find the volume of a pyramid if it is on a triangular base with sides of 7″, 24″ and 25″ and has an altitude of 27″.

6. A pyramid on a square base 10″ to the side has a slant height of 13″. Calculate (*a*) its altitude, (*b*) its volume.
7. Find the volume of a hexagonal pyramid with each side of the base 20″ and an altitude of 40″.
8. Find the volume of a pyramid on a hexagonal base 4″ to the side if the height of the pyramid is 10″.
9. If a pyramid contains 50 cu. ft. and the area of its base is 25 sq. ft., find its height.
10. A pyramid on a square base has a volume of 2,700 cu. in. If its height is 25 in., find the length of one side of the base.

The Area of a Sphere

The area of the surface of a sphere can be shown to be equal to the area of the curved surface of a cylinder of the same diameter and height as the sphere.



A Sphere and Its Corresponding Cylinder

Let the area of the sphere in sq. units be A.
Let the diameter of the sphere and the cylinder in linear units be d.
Then, the height of the corresponding cylinder in linear units is d.
Then, the area of the curved surface of cylinder in sq. units is $\pi d \times d$.
Hence, the area of the sphere in sq. units is πd^2.
Therefore, $A = \pi d^2$.
If r is the measure of the radius of the sphere this relationship may be expressed by the formula $A = 4\pi r^2$.

Example: Calculate the area of a sphere of diameter 3″.
Solution: The area relationship is $A = \pi d^2$.
The area of the sphere in sq. in. is $3{\cdot}14 \times 3^2 = 28{\cdot}3$.

Exercise

B

1. Calculate the area of the spheres the radii of which are as follows:
   (*a*) 3″, (*b*) 6″, (*c*) 5″, (*d*) 10″, (*e*) 2·5″, (*f*) 7·5″.
2. A cylinder has a diameter of 10″ and a height of 10″ inside measurements. What is the area of a sphere that will fit just inside this cylinder?
3. A sphere has a diameter of 8″. Calculate its area. Compare the area of the sphere with the area of the great circle of this sphere. Any cross-section of a sphere that passes through the centre is a **great circle**.
4. Calculate the entire area of a hemisphere if the diameter is 28″. Use $\pi = \frac{22}{7}$.
5. A closed tank is in the form of a cylinder with hemispherical ends. If the cylindrical portion is 5′ long and has a diameter of 3′, calculate the entire outside area.
6. Calculate the entire area of the surface of a hemisphere if its diameter is 15″.
7. A closed tank is in the form of a cylinder with hemispherical ends. If the over-all length of the tank is 18 feet and its diameter is 3 feet, calculate its outside area.
8. A small circle of a 10″ sphere is obtained by cutting the sphere through a point 3″ from the centre. Calculate
   (*a*) the radius of the small circle, (*b*) the area of the small circle.
9. A sphere has a diameter of 50″. Calculate the area of a small circle which is 7″ from the centre of the sphere.
10. A small circle of a 26″ sphere has a diameter of 24″. How far from the centre of the sphere is this small circle?

The Volume of a Sphere

The volume of a sphere can be shown to be equal to $\frac{2}{3}$ of the volume of the cylinder of the same diameter and height as the sphere.

Let the volume of the sphere in cu. units be V.

Let the diameter of the sphere and the cylinder in linear units be d.

Then, the height of the corresponding cylinder in linear units is d.

Then, the volume of the cylinder in cu. units is $\frac{\pi d^2}{4} \times d$.

Hence, the volume of the sphere in cu. units is $\frac{2}{3} \times \frac{\pi d^2}{4} \times d = \frac{\pi d^3}{6}$.

Therefore, $V = \frac{\pi d^3}{6}$.

If the radius of the sphere is r units the volume relationship may be expressed by the formula $V = \frac{4\pi r^3}{3}$.

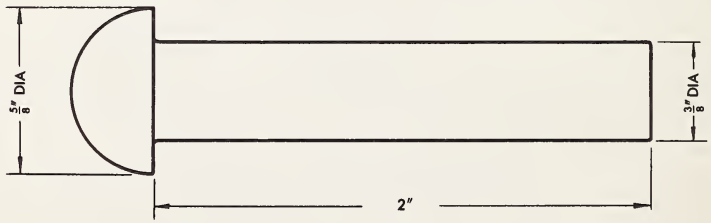
Exercise

B

1. Calculate the volumes of the spheres which have diameters as follows:
   (*a*) 1″, (*b*) 2″, (*c*) 3″, (*d*) 9″.
2. (*a*) Find the volume of a cylinder with the diameter and the height each equal to 10″.
   (*b*) Find the volume of the sphere that will fit just inside the cylinder of part (*a*).
3. Find the weight of 1,000 lead B.B. shots if each is $\frac{1}{8}$″ in diameter. Lead weighs 0·409 lb. per cubic inch.
4. If a lead ball is $3\frac{3}{4}$″ in diameter, calculate its weight.
5. Find the volume of a hemisphere 3·5″ in diameter.
6. Find the difference in volume between a cube 8″ to the side and a sphere 8″ in diameter.
7. A cylindrical storage tank has hemispherical ends. If its diameter is 6′ and its over-all length is 24′, calculate its capacity in gallons.
8. Find the ratio of the volumes of two spheres, the diameters of which are as follows:
   (*a*) 5″ and $2\frac{1}{2}$″, (*b*) 12″ and 4″, (*c*) 1″ and $\frac{1}{8}$″.

C

9. Find the weight of 1,000 button head rivets of the type and dimensions shown. They are made of steel weighing 0·285 lb. per cubic inch.



Review Exercise

B

1. A cone has a slant height of 24″ and a diameter of 16″. Determine
   (*a*) the altitude of the cone; (*b*) the lateral area of the cone;
   (*c*) the volume of the cone.
2. A pyramid 12″ high is built on a square base with each side 12″. Determine
   (*a*) the slant height of the pyramid;
   (*b*) the total area of the pyramid;
   (*c*) the volume of the pyramid.
3. A sphere has a diameter of 3″. Calculate
   (*a*) its surface area, (*b*) its volume.
4. A grapefruit is spherical in shape and has a skin of $\frac{1}{2}$″ thickness. If the grapefruit is 6″ in diameter with skin on, what is the actual volume of the edible fruit?
5. If the skin in problem 4 were just $\frac{1}{4}$″ thick, how many extra cubic inches of edible fruit would there be?
6. An orange with outside diameter of 4″ has a skin $\frac{1}{4}$″ thick. Find the volume of edible fruit in a hemisphere of orange.

CHAPTER 22

Trigonometry

Triangle Relationships

A few relationships between the lengths of the sides of a triangle and the sizes of the angles have already been noted. For example:

(i) If two sides of a triangle are equal, the angles opposite these sides are equal to each other.

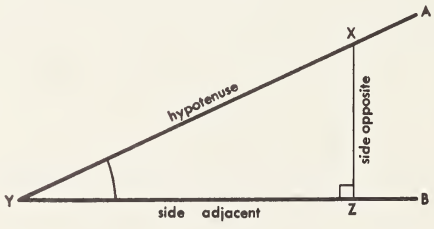
(ii) If three sides of a triangle are equal, each angle is 60°.

(iii) If the square on one side of a triangle equals the sum of the squares on the other two sides, then the angle opposite the first side is a right angle.

The first relationship does not enable us to obtain the measure of the angles but the second and third do.

In this chapter we shall establish relationships between the lengths of the sides of a right-angled triangle and the sizes of its acute angles. This study is usually referred to as Trigonometry.

The Definitions of the Principal Trigonometrical Ratios in a Right-angled Triangle



Let **AYB** be any acute angle and **X** any point on **AY**.

From **X** draw **XZ** perpendicular to **YB** to form the right-angled triangle **XYZ**.

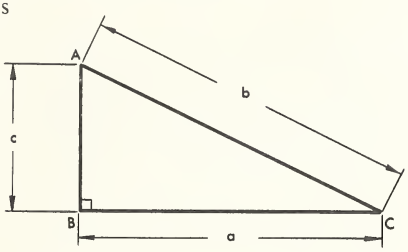
In the right-angled triangle **XYZ**, **XY** is the hypotenuse. If we consider the acute angle **Y**, **YZ** is the side adjacent to angle **Y** and **XZ** is the side opposite to angle **Y**.

The ratio $\frac{\mathbf{XZ}}{\mathbf{XY}}$ is the **sine** of angle **Y**.

The ratio $\frac{\mathbf{YZ}}{\mathbf{XY}}$ is the **cosine** of angle **Y**.

The ratio $\frac{\mathbf{XZ}}{\mathbf{YZ}}$ is the **tangent** of angle **Y**.

The three principal trigonometric ratios of $\angle \mathbf{C}$ in the triangle **ABC** may be stated as



Sine $C = \frac{\text{side opposite}}{\text{hypotenuse}}$	more briefly	$\sin C = \frac{c}{b}$
Cosine $C = \frac{\text{side adjacent}}{\text{hypotenuse}}$	more briefly	$\cos C = \frac{a}{b}$
Tangent $C = \frac{\text{side opposite}}{\text{side adjacent}}$	more briefly	$\tan C = \frac{c}{a}$

Exercise

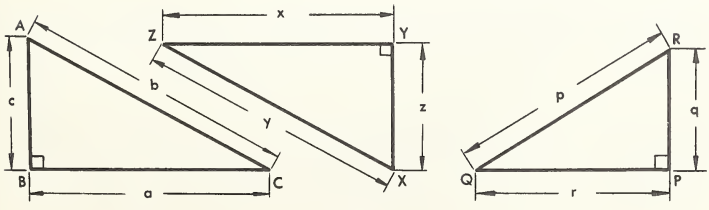
A

1. For each of the right-angled triangles illustrated, express

(*a*) the sine of each of the acute angles;

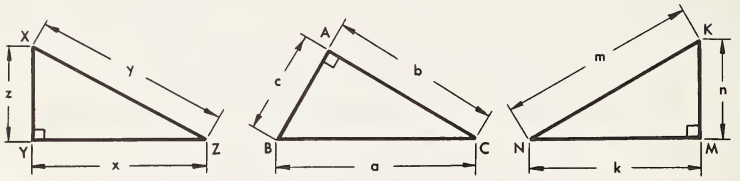
(*b*) the cosine of each of the acute angles;

(*c*) the tangent of each of the acute angles.



2. Refer to the diagrams below and state each of the following as a trigonometric ratio in two ways:

(*a*) $\frac{x}{y}$, (*b*) $\frac{c}{a}$, (*c*) $\frac{k}{m}$, (*d*) $\frac{z}{y}$, (*e*) $\frac{b}{a}$, (*f*) $\frac{n}{m}$.



3. In the diagrams above, express

(*a*) tan *Z*, tan *B*, tan *N*.

(*b*) Express as functions of acute angles $\frac{n}{m}$, $\frac{c}{b}$, $\frac{z}{x}$, $\frac{b}{c}$.

B

4. In a right-angled triangle **ABC**, the hypotenuse **AC** is 10″, the side **AB** is 8″.

(*a*) Calculate the length of **BC**.

(*b*) Find the value of each of the following ratios:

(i) sin *A*, (ii) cos *A*, (iii) tan *A*,
(iv) sin *C*, (v) cos *C*, (vi) tan *C*.

5. In the right-angled triangle **DEF**, $\angle \mathbf{E} = 90°$, **DE** is 7″ in length. **EF** is 24″ in length. Find the arithmetical value of each of the three principal trigonometrical ratios of (i) $\angle \mathbf{D}$, (ii) $\angle \mathbf{F}$.

6. Draw a right-angled isosceles triangle **ABC** with the right angle at **B** and **AB** = 1″. State:

(*a*) the size of each acute angle, (*b*) the length of **AC**, (*c*) sin 45°.

7. Draw an equilateral triangle **ABC** with each side 2″ in length. Draw **AD** perpendicular to **BC** to meet **BC** at **D**.

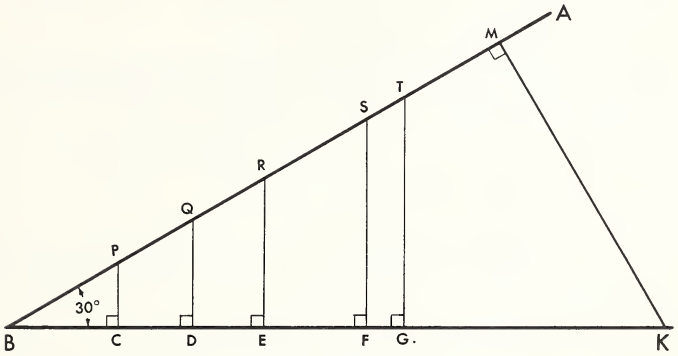
(*a*) Name and give the size of each angle in the diagram.

(*b*) Calculate the length of **AD**.

(*c*) Calculate (i) sin 30°, (ii) sin 60°.

C

8. (*a*) Construct $\angle \mathbf{B} = 30°$ and on one arm choose points **P**, **Q**, **R**, **S** and **T**. From these points draw perpendiculars to the other arm to cut the other arm at **C**, **D**, **E**, **F** and **G** respectively. Make the necessary measurements and calculate the value of sin *B* using (i) triangle **PBC**, (ii) triangle **QBD**, (iii) triangle **RBE**, (iv) triangle **SBF**, (v) triangle **TBG**.



(*b*) Select any point **K** on **BC**. From **K** draw a perpendicular to **AB** to intersect it at **M**. Using the triangle **KBM** find sin *B*.

(*c*) Compare the values of the sine ratio of 30° obtained from the different right-angled triangles.

9. Draw any acute angle **DEF** and along **DE** choose points **A**, **B** and **C** at distances of 1″, 2″ and 3″ respectively from the vertex **E**. From each of these points draw perpendiculars to **EF**. From each right-angled triangle calculate sin *E*.

These calculations illustrate that the sine of an angle is constant, regardless of the triangle from which it is calculated.

10. (*a*) Using the diagram constructed for problem 8, make the necessary measurements and find the cosine ratio of the angle **B** from each of the triangles.

(*b*) Using the diagram constructed for problem 9 above, find the cosine ratio of angle **E** from each triangle.

State a conclusion regarding the cosine of an angle.

11. (*a*) Using the diagram constructed for problem 8, find the tangent ratio of the angle **B** from each of the triangles.

(*b*) Using the diagram constructed for problem 9 above, find the tangent ratio of the angle **E** for each triangle.

State a conclusion regarding the tangent of an angle.

Tables

The usual method of obtaining trigonometrical ratios is to refer to prepared tables of values. On page 395, a table is given which shows each angle from 0° to 90° in intervals of one degree. Opposite each angle is listed the value of the three principal ratios.

Example 1: Find sin 35°.

Solution: By reference to the table sin 35° = 0·5736.

Example 2: Given sin $A = 0{\cdot}8498$, find $\angle A$.

Solution: From the column of sine ratios in the table, it is found that 0·8498 lies between 0·8572 and 0·8480 but it is closer to 0·8480.
Since 0·8480 is sin 58°,
therefore, $\angle A = 58°$ approximately.

Example 3: If cos $A = 0{\cdot}4633$, find angle A.

Solution: Since cos 62° = 0·4695 and
cos 63° = 0·4540 therefore A lies between 62° and 63° but it is closer to 62°.

$\therefore \angle A = 62°$ approximately.

Exercise

A

1. Find from the tables, the values of each of the following:
(*a*) sin 40°, (*b*) sin 25°, (*c*) sin 75°, (*d*) sin 89°.

2. Use the table of sines to find, to the nearest degree, the angles when
(*a*) sin $A = 0{\cdot}2756$, (*b*) sin $B = 0{\cdot}8212$, (*c*) sin $C = 0{\cdot}8262$.

3. Use the table of cosines to determine
(*a*) cos 28°, (*b*) cos 55°, (*c*) cos 74°,
(*d*) cos 3°, (*e*) cos 12°, (*f*) cos 30°.

4. Use the table of cosines to determine to the nearest degree the angle indicated when
(*a*) cos $A = 0{\cdot}3090$; (*b*) cos $B = 0{\cdot}9455$; (*c*) cos $C = 0{\cdot}7988$;
(*d*) cos $X = 0{\cdot}8488$; (*e*) cos $B = 0{\cdot}7600$; (*f*) cos $Z = 0{\cdot}3650$.

5. Using the table, find the value of
(*a*) tan 12°, (*b*) tan 67°, (*c*) tan 56°, (*d*) tan 67°.

6. Find the angles which satisfy the following equations:
(*a*) tan $X = 0{\cdot}3839$; (*b*) tan $Y = 2{\cdot}9042$; (*c*) tan $K = 1$;
(*d*) tan $A = 1{\cdot}732$; (*e*) tan $B = 1{\cdot}8977$; (*f*) tan $C = 1\frac{7}{8}$.

B

7. Find, to the nearest degree, the size of the angle when

(*a*) $\sin A = 0{\cdot}4226$; (*b*) $\sin B = 0{\cdot}9013$; (*c*) $\sin C = 0{\cdot}6111$;
(*d*) $\sin D = 0{\cdot}7822$; (*e*) $\sin E = \frac{3}{4}$; (*f*) $\sin F = \frac{7}{8}$.

8. Solve the following equations for x:

(*a*) $\frac{x}{10} = \sin 33°$; (*b*) $\frac{x}{24} = \sin 55°$; (*c*) $\frac{10}{x} = \sin 30°$.

9. Find the angles which satisfy the following equations:

(*a*) $\cos A = 0{\cdot}25$; (*b*) $\cos Y = 0{\cdot}333$; (*c*) $\cos C = 0{\cdot}88$;
(*d*) $\cos K = \frac{3}{4}$; (*e*) $\cos M = \frac{7}{8}$; (*f*) $\cos N = 1{\cdot}0$.

10. Solve the following equations for x:

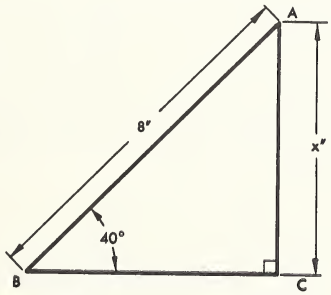
(*a*) $\frac{x}{5} = \cos 60°$; (*b*) $\frac{x}{30} = \cos 15°$; (*c*) $\frac{20}{x} = \cos 60°$.

11. Solve for x in each of the following equations:

(*a*) $\frac{x}{5} = \tan 45°$; (*b*) $\frac{x}{3{\cdot}6} = \tan 38°$; (*c*) $\frac{10}{x} = \tan 77°$.

The Use of the Sine Ratio to Find the Length of the Side of a Right-angled Triangle

Example: In the triangle **ABC**, $\angle \mathbf{C} = 90°$, **AB** $= 8{\cdot}0''$ and $\angle \mathbf{B} = 40°$. Calculate **AC**.



Solution: Let the length of **AC** in inches be x.

From the triangle $\frac{x}{8} = \sin 40°$.

From the table $\sin 40° = 0{\cdot}6428$.

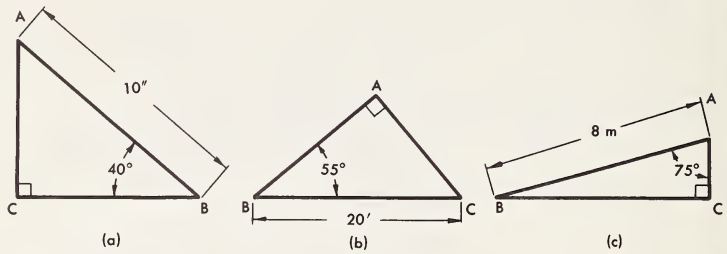
Therefore, $\frac{x}{8} = 0{\cdot}6428$.

$x = 5{\cdot}1424$.

The length of **AC** is $5{\cdot}1''$.

Exercise B

1. Calculate the length of the side **AC** in each of the following right-angled triangles:



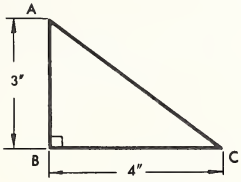
2. For each of the following right-angled triangles draw a sketch showing the given dimensions and calculate the measure of each of the remaining parts of the triangle:
   (*a*) Triangle **ABC** has $\angle \mathbf{B} = 90°$, $\angle \mathbf{C} = 48°$ and the length of **AC** $= 25''$.
   (*b*) Triangle **DEF** has $\angle \mathbf{E} = 90°$, $\angle \mathbf{D} = 75°$ and **DF** $= 10''$.
   (*c*) Triangle **GHK** has $\angle \mathbf{H} = 90°$, $\angle \mathbf{G} = 54°$ and **GK** $= 3{\cdot}6''$.
3. If a ladder 15′ long is mounted on a truck, to what height above the truck platform will the ladder reach when elevated to an angle of 80°?
4. If the grade of a hill on the highway is 4°, what distance, in feet, has a car raised itself when it has travelled $\frac{1}{2}$ mile up the road?
5. If the slant height of a cone is 12″ and it makes an angle of 63° with the base, calculate the height of the cone.
6. Calculate the side of a square which can be inscribed in a circle of diameter (*a*) 10″, (*b*) 6″, (*c*) $\frac{7}{8}''$.
7. In an isosceles triangle, each of the equal sides is 20″ and the angle contained by them is 100°. Calculate the length of the third side.
8. If a regular pentagon is inscribed in a circle of radius 4″, calculate the length of its side.
9. Calculate the depth of a sharp V thread if the pitch of the thread is given as (*a*) $\frac{1}{8}''$, (*b*) $\frac{1}{12}''$, (*c*) $\frac{1}{40}''$.
10. If a chord of an 8″ circle subtends an angle of 66° at the centre of the circle, determine the length of the chord.
11. An aeroplane is travelling a course N 40° E at a ground speed of 200 m.p.h. How far north from its starting-point is it at the end of $\frac{3}{4}$ of an hour?

C

12. In the triangle **ABC**, **AB** = 10″, ∠**B** = 40° and **BC** = 15″. Calculate
(*a*) the length of the altitude **AD** from **A** to **BC**;
(*b*) the length of **BD**; (*c*) the length of **DC**;
(*d*) the length of **AC** to the nearest tenth of an inch;
(*e*) the size of ∠**C**; (*f*) the size of ∠**BAC**.

The Use of the Sine Ratio to Find the Size of an Acute Angle of a Right-angled Triangle

Example 1: The sides of a right-angled triangle are 3″ and 4″ respectively. Solve for the other three principal parts of this triangle.



Solution:

In the triangle **ABC** let ∠**B** = 90°, **AB** = 3″ and **BC** = 4″.

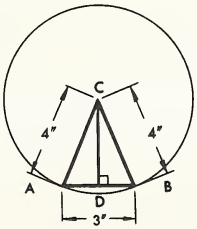
The length of **AC** in inches is $\sqrt{3^2 + 4^2} = 5$.

From the triangle, $\sin A = \frac{4}{5}$ $\sin C = \frac{3}{5}$

$= 0{\cdot}80$ $= 0{\cdot}60$

From the table $\angle A = 53°$ approx. $\angle C = 37°$ approx.

Example 2: A chord 3″ long is drawn in a circle with a diameter of 8″. Find the size of the angle that the chord subtends at the centre of the circle.



Solution:

In the diagram, draw **CD** perpendicular to **AB** to meet **AB** at **D**.

In the right-angled triangle **ADC**, **AD** = $1\frac{1}{2}$″.

$$\sin \mathbf{ACD} = \frac{1{\cdot}5}{4}.$$

$\sin \mathbf{ACD} = 0{\cdot}375.$

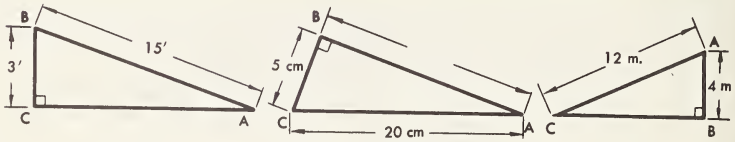
Therefore, $\angle \mathbf{ACD} = 22°$ to the nearest degree.

Therefore, $\angle \mathbf{ACB} = 44°$ approx.

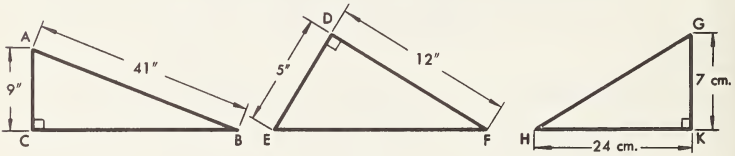
Exercise

B

1. Calculate the size of $\angle \mathbf{A}$ in each of the following right-angled triangles:



2. Calculate the acute angles in each of the following triangles:



3. Calculate the angle subtended at the centre of a 4″ circle by a $3\frac{1}{2}$″ chord.

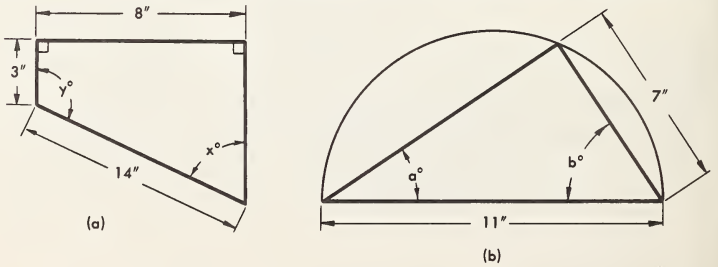
4. If a straight road rises 200 feet in a distance of $\frac{1}{2}$ mile along the road, calculate the angle of slope that the road makes with the horizontal.

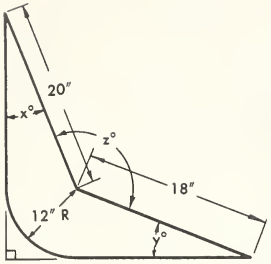
5. Calculate the angles of the isosceles triangles with lengths of sides as follows:

(*a*) each of the equal sides is 25″ and the base is $26\frac{1}{2}$″;
(*b*) each of the equal sides is 3·2″ and the base is 1″.

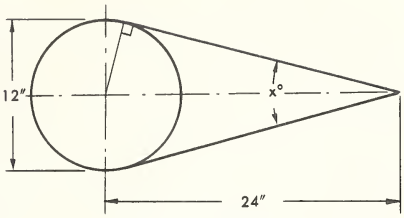
6. If a cone has a slant height of 17″ and a base diameter 16″, calculate the angle that the slant height makes with the base.

7. Determine the sizes of the angles marked in the following diagrams:





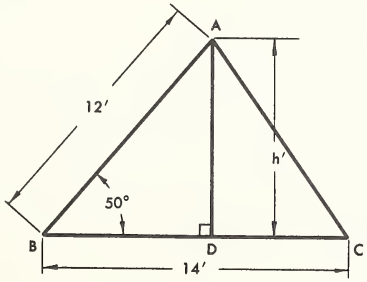
(c)



(d)

The Area of a Triangle when Two Sides and the Contained Angle are Known

Example: In the triangle **ABC**, the length of **AB** is 12 feet, the length of **BC** is 14 feet and $\angle \mathbf{B} = 50°$. Calculate the area of the triangle **ABC**.



Solution:

Draw **AD** perpendicular to **BC** to meet **BC** at **D**.

Let the length of **AD** in feet be h.

In the triangle **ABD**, $\angle \mathbf{ADB} = 90°$, $\sin 50° = \dfrac{h}{12}$.

From the table $\sin 50 = 0{\cdot}7660.$

$$\frac{h}{12} = 0{\cdot}7660.$$

$$h = 9{\cdot}192.$$

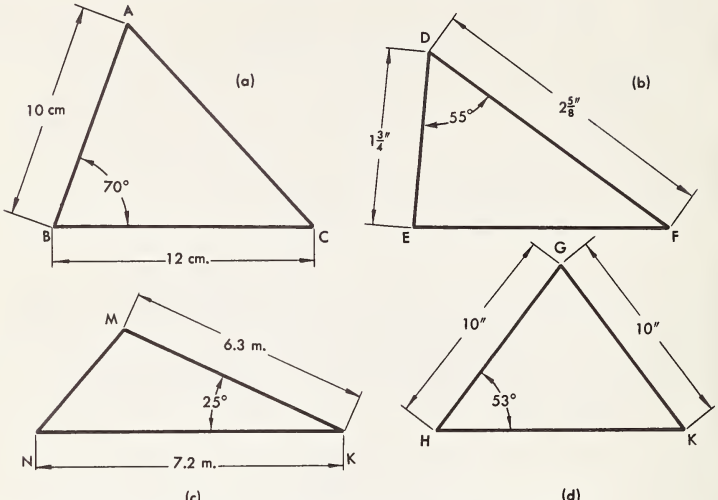
The area of the triangle in sq. ft. is $\frac{1}{2} \times 14 \times 9{\cdot}192 = 64{\cdot}3$.

Exercise

B

1. Calculate the area of the triangle **ABC** in which the length of **AB** is 5″, $\angle \mathbf{B} = 49°$ and the length of **BC** is 6″.

2. Calculate the area of each of the following triangles:



3. Calculate the area of the following obtuse-angled triangles:

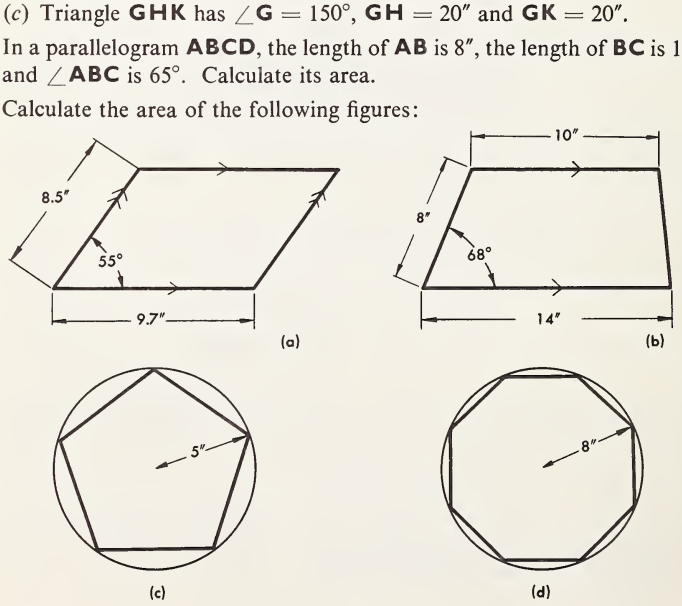
(*a*) Triangle **ABC** has $\angle \mathbf{B} = 120°$, **AB** $= 4''$ and **BC** $= 12''$;

(*b*) Triangle **DEF** has $\angle \mathbf{F} = 110°$, **DF** $= 16{\cdot}4''$ and **EF** $= 12{\cdot}8''$;

(*c*) Triangle **GHK** has $\angle \mathbf{G} = 150°$, **GH** $= 20''$ and **GK** $= 20''$.

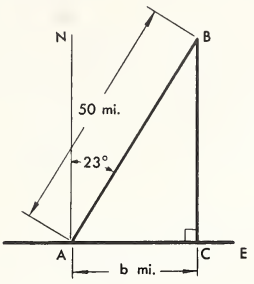
4. In a parallelogram **ABCD**, the length of **AB** is $8''$, the length of **BC** is $10''$ and $\angle$**ABC** is 65°. Calculate its area.

5. Calculate the area of the following figures:



The Use of the Cosine Ratio to Find the Length of the Side of a Right-angled Triangle

Example: A certain town **B** is 50 miles N 23° E of town **A**. How far east of **A** is **B**?



Solution:

Let the distance that **B** is east of **A** in miles be b.

In the triangle **ACB**, $\angle \mathbf{C} = 90°$, $\dfrac{b}{50} = \cos 67°$.

From the table $\cos 67° = 0{\cdot}3907$.

$$\frac{b}{50} = 0{\cdot}3907.$$

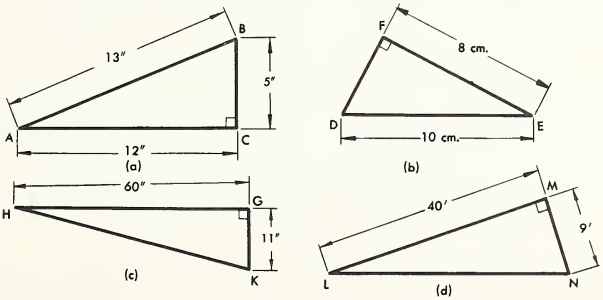
$$b = 19{\cdot}5.$$

Town **B** is approximately 19·5 miles east of town **A**.

Exercise

A

1. For the following triangles state the cosine ratio of each acute angle:



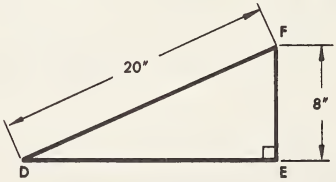
B

2. If a ladder 36′ long leans against a wall at an angle of 66°, calculate the distance from the foot of the ladder to the wall.

3. If the side of a rectangle makes an angle of 50° with the diagonal which is 16″ long, calculate the length of the side.
4. Calculate the side of a square if the diagonal is
   (*a*) 10″, (*b*) 4″, (*c*) 8·5″, (*d*) 23·45 cm., (*e*) 127·6′.
5. Calculate the size of the largest square that can be milled from round stock of diameter (*a*) $\frac{1}{2}$″, (*b*) $\frac{3}{4}$″, (*c*) 2″.
6. Two equal lines **AB** and **AC** each 12″ long are drawn to meet at an angle of 32° at **A**. Calculate the distance **BC**.
7. Calculate the length of a side of a regular pentagon inscribed in a circle of radius 8″.
8. If cos $67\frac{1}{2}° = 0{\cdot}3827$, calculate the length of the side of a regular octagon inscribed in a circle of diameter 24″.
9. The vertical angle of a cone is 100° and the slant height is 16′. Calculate the vertical height.
10. If a chord of a 12″ circle makes an angle of 33° with a diameter at the circumference, calculate the length of the chord.
11. In the triangle **ABC**, **AB** = 6″, **BC** = 8″ and $\angle$**B** = 73°. If **AD** is drawn perpendicular to **BC**, calculate the distance **BD**. **BD** is the **projection** of **AB** on **BC**.
12. In the triangle **ABC**, **AB** = 7·5″, **BC** = 12″ and $\angle$**B** = 39°. Calculate the projection of **AB** on **BC** and also of **AC** on **BC**.
13. If a man walks N 60° W for 8 miles, how far west of his starting-point will he be?
14. To avoid a shoal, a ship on an easterly course sails N 66° E for 20 miles and S 66° E for 20 miles and then continues on its course. How many extra miles did the ship sail because of the shoal?
15. If the 10-inch diagonal of a rectangle makes an angle of 35° with one of its sides, calculate the length and width of the rectangle.

The Use of the Cosine Ratio to Find the Size of an Acute Angle in a Right-angled Triangle

Example: In the triangle **DEF**, $\angle$**E** = 90°, the length of **DF** is 20″ and the length of **EF** is 8″. Solve for the other parts of this triangle.



Solution:

From the triangle, $\cos F = \frac{8}{20}$

$\cos F = 0{\cdot}4.$

From the tables, $\angle \mathbf{F} = 66°$ approximately.

$\angle \mathbf{D} = 90° - 66° = 24°$ approx.

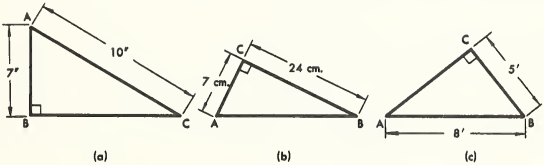
The length of **DE** may be found by any one of three methods

$\frac{\mathbf{DE}}{20} = \cos 24°$ or $\frac{\mathbf{DE}}{20} = \sin 66°$ or $\mathbf{DE} = \sqrt{20^2 - 8^2}.$

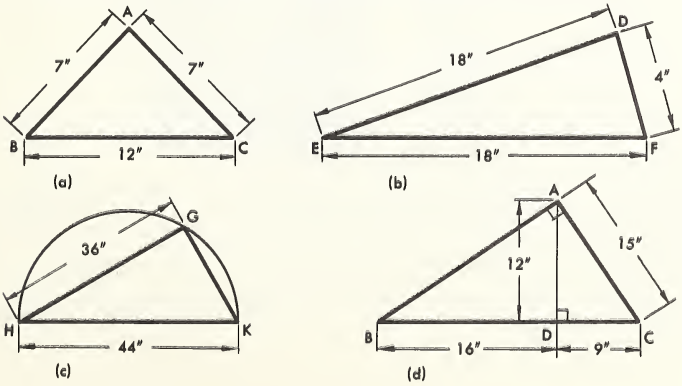
The length of **DE** in inches is 18·3.

Exercise B

1. Use the cosine ratio to calculate $\angle \mathbf{A}$ in each of the following triangles:



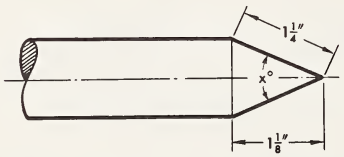
2. A 24′ ladder, leaning against a wall, has its foot 6′ 6″ from the wall. Find the angle the ladder makes with the ground.
3. The diagonal of a rectangle is 12″ long and the length of one side is 10·5″. Find the angle that the diagonal makes with this side.
4. If the sides of an isosceles triangle are 20″, 20″ and 13″, calculate the sizes of its angles.
5. Calculate each of the acute angles in the following diagrams:

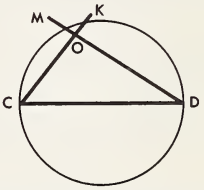


6. If a cone has an altitude of 28″ and a slant height of 32″, calculate the size of its vertical angle.
7. If the radius of the earth is 4,000 miles, calculate
   (*a*) the radius of the small circle at latitude 45°;
   (*b*) the distance that a man would travel if he circled the globe along the 45° parallel of latitude.

C

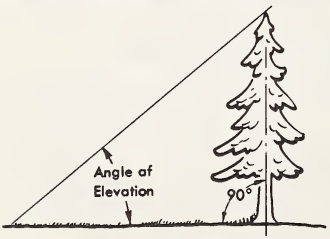
8. If cos X = sin 60°, what is the size of $\angle$**X**?
9. If cos Y = sin 28°, what is the size of $\angle$**Y**?
10. How could you find the cosine of an angle from the sine table?
11. A point **A** is marked 7·5 inches above a horizontal line **XY**. With centre **A** and radius 12″, a point **C** is marked on **XY**. With centre **A** and radius 16″, another point **D** is marked on **XY**. Calculate $\angle$**CAD**. Note: There are two solutions.
12. Calculate angle x in the diagram below.

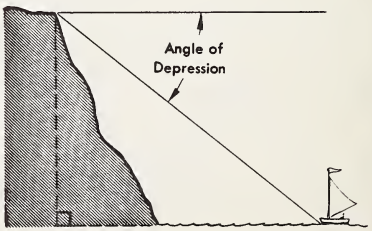




13. In the circle above, the diameter **CD** is 30″, the chord **CK** is 24″ and the chord **DM** is 20″. Calculate
   (*a*) $\angle$**OCD**, (*b*) $\angle$**ODC**, (*c*) $\angle$**KOM**.

Angles of Elevation and Depression



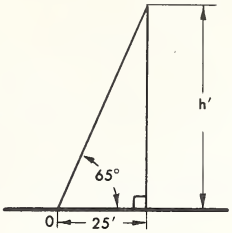


The **angle of elevation** is the angle through which the line of sight must be raised above the horizontal in order to see a distant point which is above the observer's eye level.

The **angle of depression** is the angle through which the line of sight must be lowered below the horizontal in order to see a distant point which is below the observer's eye level.

The Use of the Tangent Ratio to Find the Length of the Side of a Right-angled Triangle

Example 1: The angle of elevation of the top of a flag-pole is taken by an observer standing 25 feet from the pole and is found to be 65°. Calculate the height of the pole.



Solutions:

Let the height of the pole above eye level in feet be h.

From the triangle $\frac{h}{25} = \tan 65°.$

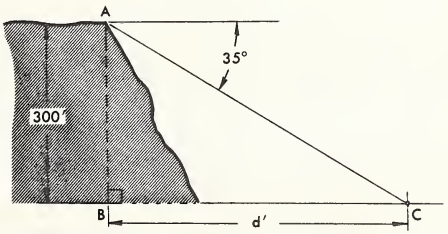
From the table $\tan 65° = 2{\cdot}1445.$

$$\frac{h}{25} = 2{\cdot}1445.$$

$$h = 53{\cdot}6.$$

Therefore, the height of the pole above the point of observation is 53·6 feet.

Example 2: An observer on a cliff 300 feet high observes a boat at an angle of depression of 35°. How far is the boat from the base of the cliff?



Solution:

Let the distance from the boat to the cliff in feet be d.

In the diagram $\angle$**BAC** in degrees is $90 - 35 = 55$.

From the triangle $\frac{d}{300} = \tan 55°.$

From the table $\tan 55° = 1{\cdot}4281.$

$$\frac{d}{300} = 1{\cdot}4281.$$

$$d = 428{\cdot}4.$$

Therefore, the boat is 428 feet from the cliff.

Exercise

B

1. From a point 100 feet from the foot of a flag-pole, the angle of elevation of the top of the pole is 55°. Calculate the height of the pole.
2. From a point 450 feet from the foot of a high building, the angle of elevation of the top of the building is 42°. Calculate the height of the building.
3. From the top of a fire tower 300 feet high, a fire is observed at an angle of depression of 8°. How far is the fire from the base of the tower?
4. If one side of a rectangle is 18″ and the diagonal makes an angle of 30° with this side, what is the length of the other side?
5. The base of an isosceles triangle is 24″ and the base angles are each 70°. Find the altitude.
6. When the angle of elevation of the sun is 23°, what will be the length of the shadow cast by a building 45 feet high?
7. From the top of a cliff 200 ft. above sea-level, a small sailing vessel is observed at an angle of depression of 15°. How far is the vessel from the foot of the cliff?
8. Using the tangent ratio, calculate the height of each of the following equilateral triangles:
   (*a*) each side 12″, (*b*) each side $\frac{1}{8}$″, (*c*) each side $\frac{1}{40}$″.
9. Calculate the depth of a National Coarse thread which has 10 threads per inch.
10. Calculate the distance across the flats of a hexagon which has each side equal to (*a*) 6″, (*b*) $\frac{1}{2}$″.
11. If the side of a regular pentagon is 5″, calculate its area.

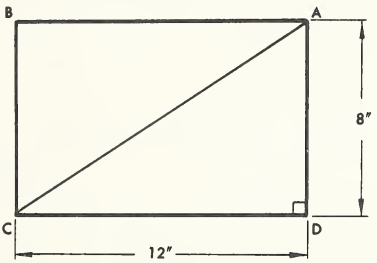
C

12. If a square 16″ to a side is cut across the corners to form a regular octagon, what is the length of each side of the octagon?
13. Calculate the length of each side of a regular octagon which can be cut from shelving board $11\frac{1}{2}$″ wide.
14. In triangle **ABC**, $\angle \mathbf{B} = 23°$, $\angle \mathbf{C} = 47°$ and the altitude from **A** to **BC** is 84″. Calculate the length of **BC**.

15. From a point 150 ft. from the base of a building, the angle of elevation to the top of the building is 32° and the angle of elevation of the top of a flag-pole on the top of the edge of the building is 42°. Calculate the height of the pole above the top of the building.
16. Calculate the radius of the inscribed circle of an equilateral triangle if each side is 5″.
17. In an isosceles triangle, the angles at the base are each 56° and the length of the base is 18″. Calculate the radius of the inscribed circle.

The Use of the Tangent Ratio to Find the Size of an Angle of a Right-angled Triangle

Example 1: If the sides of a rectangle are 12″ and 8″ respectively, find the size of the angle that the diagonal makes with the 12″ side.



Solution:

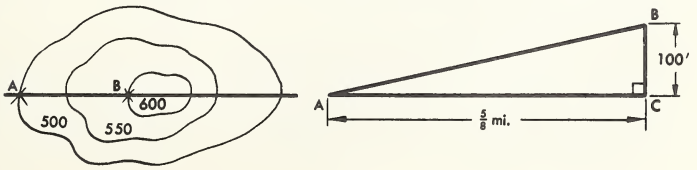
In the triangle **ACD** $\tan \mathbf{ACD} = \frac{8}{12}$

$= 0{\cdot}667.$

From the table $\angle \mathbf{ACD} = 34°.$

Therefore, the diagonal makes an angle of 34° with the 12″ side.

Example 2: A map shows a straight road crossing two contour levels of 500 and 600 feet at **A** and **B** respectively. If the distance between **A** and **B** on the map is $\frac{5}{8}$″ and the scale of the map is 1 inch = 1 mile, calculate the angle that the road makes with the horizontal.



Solution:

The length of **AC** in feet is $\frac{5}{8} \times 5{,}280 = 3{,}300$.

From the triangle $\tan A = \dfrac{100}{3{,}300}$

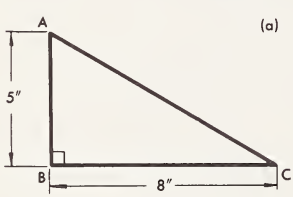
$= 0{\cdot}0303.$

From the table $\angle \mathbf{A} = 2°$.

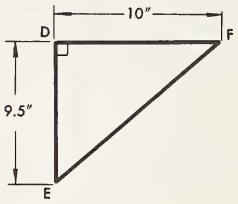
The angle that the road makes with the horizontal is 2°.

Exercise B

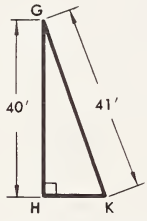
1. If the sides of a rectangle are 6″ and 8″, calculate the angles that the diagonal makes with the sides.
2. Calculate the acute angles in each of the following right-angled triangles:



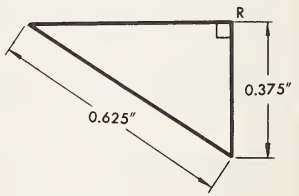
(a)



(b)



(c)



(d)

3. If the shadow of a yardstick is measured to be 24″, calculate the angle of elevation of the sun above the horizon.
4. If the diagonals of a rectangle 30′ by 21′ are drawn, at what angle will they intersect?
5. If the height of a cone is 15″ and the diameter of the base is 14″, calculate the vertical angle of the cone and also the angle that the slant height makes with the base.
6. If a ladder is placed with its base 8 feet from the wall of a house and it reaches 35 feet up the wall, what is the angle of elevation of the ladder?
7. In any rhombus, the diagonals bisect each other at right angles. Calculate the angles of a rhombus if the diagonals are 28″ and 14″ respectively.

8. A map shows a straight road crossing two contour levels of 525 feet and 625 feet at **X** and **Y** respectively. If the distance **XY** on the map is $\frac{3}{8}''$ and the scale of the map is 1 inch = 1 mile, calculate the angle that this road makes with the horizontal.

9. A map shows a hill between contour levels of 800 and 1,025 feet with a distance on the map of $\frac{3}{8}''$. If the scale of the map is 1 inch = 1 mile, find the angle of the slope of the hill.

10. From a point 100 feet from the foot of a flag-pole, the angle of elevation of the top of the pole is 56°. Find the angle of elevation of a point half-way up the pole from the same point on the ground.

11. The treads of a stair are 10″ and the rise of each step is $7\frac{1}{2}''$. What is the angle of elevation of the stairs?

C

12. Calculate, by division, the value of $\frac{1}{\tan 25^\circ}$ correct to three places of decimals and compare it with the value of tan 65°.

13. Find, by division, the value of $\frac{1}{\tan 30^\circ}$ and compare it with tan 60°.

14. Find the value of (*a*) $\frac{1}{\tan 10^\circ}$, (*b*) $\frac{1}{\tan 84^\circ}$.

15. If a tapered piece of material is produced to form a cone, the vertical angle of the cone is the **angle of the taper**. Calculate the angle of the taper in each of the following cases:

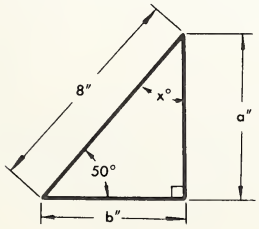
(*a*) a taper of 3·5″ per foot; (*b*) a taper of 13·86″ per foot;
(*c*) a taper of $\frac{1}{2}''$ per foot; (*d*) a taper of 0·600″ per foot;
(*e*) a taper of $\frac{5}{8}''$ per foot; (*f*) a taper of $\frac{1}{4}''$ in a length of 8″.

Tables giving angles to the nearest minute should be used.

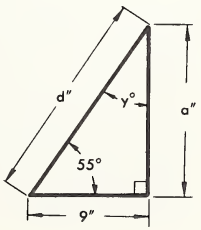
Review Exercise

B

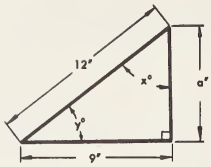
1. Using the dimensions shown on the diagrams below, calculate the measures indicated by the letters.



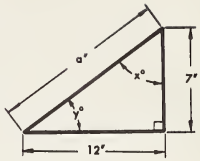
(a)



(b)

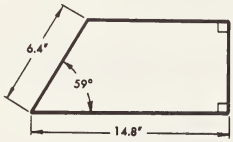


(c)

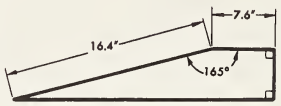


(d)

2. Solve the triangle **ABC** with $\angle \mathbf{B} = 90°$ if
   (*a*) **AC** is 42″ and $\angle \mathbf{C}$ is 81°, (*b*) **AC** is 10·8″ and $\angle \mathbf{C}$ is 40°,
   (*c*) **AC** is 1·92″ and $\angle \mathbf{A}$ is 51°.
3. Solve the triangle **ABC** with $\angle \mathbf{B} = 90°$ if
   (*a*) **BC** is 15″ and $\angle \mathbf{C}$ is 50°, (*b*) **AB** is 19″ and $\angle \mathbf{C}$ is 70°,
   (*c*) **BC** is 2·3″ and $\angle \mathbf{A}$ is 58°.
4. Calculate the areas of the following figures:



(a)



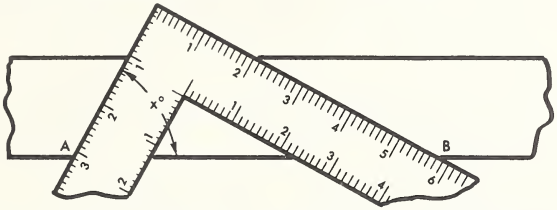
(b)

5. Solve the triangle **ABC** with $\angle \mathbf{B} = 90°$ if
   (*a*) **AC** is 25′ and **AB** is 24′, (*b*) **AC** is 33″ and **BC** is 18″,
   (*c*) **AC** is 13·9″ and **AB** is 8·5″.
6. Solve the triangle **ABC** with $\angle \mathbf{B} = 90°$ if
   (*a*) **AB** is 36″ and **BC** is 15″, (*b*) **AB** is 45″ and **BC** is 24″,
   (*c*) **AB** is 8″ and **BC** is 9″.
7. If a number of holes are equally spaced round a circle, calculate the distance between the centres of adjacent holes if there are
   (*a*) 3 holes equally spaced on a 4″ circle;
   (*b*) 4 holes equally spaced on a 6·6″ circle;
   (*c*) 12 holes equally spaced on a 10″ circle;
   (*d*) 24 holes equally spaced on a 10″ circle.
8. Find the area of each of the following triangles:
   (*a*) Triangle **ABC** has **AB** = 10·6″, **BC** = 24·2″, $\angle \mathbf{B} = 76°$;
   (*b*) Triangle **ABC** has **AB** = 7·4″, **AC** = 15·4″, $\angle \mathbf{A} = 45°$;
   (*c*) Triangle **ABC** has **AD** perpendicular to **BC**, **AD** = 8·8″, $\angle \mathbf{BAD} = 27°$, $\angle \mathbf{DAC} = 43°$;
   (*d*) Triangle **ABC** with $\angle \mathbf{B} = 90°$, $\angle \mathbf{C} = 38°$, **AC** = 9·4″.

9. Calculate the area of the following parallelograms:
(*a*) Parallelogram **ABCD** with **AB** = 12·4″, ∠**B** = 53°, **BC** = 18·2″;
(*b*) Parallelogram **ABCD** with diagonal **DB** = 20·0″, **BC** = 14·8″ and ∠**DBC** = 28°.

10. From the top of a tower 500 feet high, the angle of depression of a fire is found to be 3°. Calculate the distance of the fire from the base of the tower.

11. A steel square is placed as indicated on the diagram to make angles other than right angles with the edge of a board.

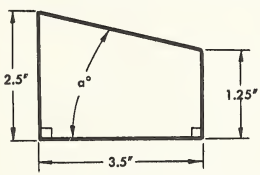


Select suitable readings, to the nearest $\frac{1}{16}$ inch, to place at **A** and **B** so that the angle marked x may be
(*a*) 45°, (*b*) 70°, (*c*) 50°, (*d*) 65°, (*e*) 35°, (*f*) 80°.

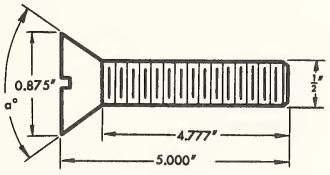
12. From a point 80 feet from the base of a tower, the angle of elevation of the top of the tower is 57°. Find the height of the tower.

13. If a pole 8′ long casts a shadow 12′ 6″, calculate the angle of elevation of the sun.

14. Calculate angle a in the diagrams below:

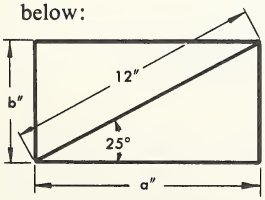


(a)

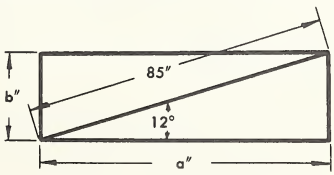


(b)

15. Calculate the measure indicated by a and b in each of the rectangles below:

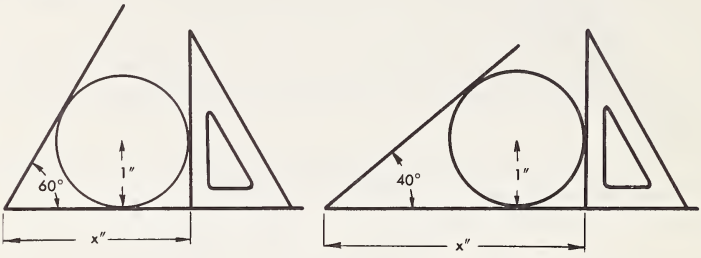


(a)



(b)

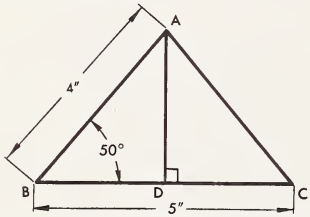
16. From the dimensions indicated on the diagrams below, calculate the measure x:

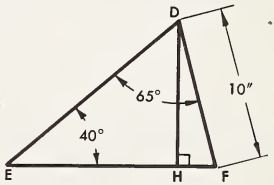


C

17. In the triangle **ABC** below, the length of **AB** is 4″, **BC** is 5″ and the contained angle **B** is 50°. Calculate

(*a*) the altitude **AD**,
(*b*) the projection of **AB** on **BC**,
(*c*) the projection of **AC** on **BC**,
(*d*) the angle **C**,
(*e*) the angle **BAC**,
(*f*) the area of triangle **ABC**.





18. In the triangle **DEF** above, the length of **DF** is 10″, $\angle \mathbf{D} = 65°$, and $\angle \mathbf{E} = 40°$. Calculate

(*a*) the angle **F**,
(*b*) the altitude **DH**,
(*c*) the length of **HF**,
(*d*) the length of **EH** and **DE**,
(*e*) the length **EF**,
(*f*) the area of triangle **DEF**.

19. In the triangle **GHK** in which the side **GH** is 13″, **GK** is 14″ and **HK** is 15″, calculate

(*a*) the area of triangle **GHK**,
(*b*) the altitude **GD** drawn to the 15″ side,
(*c*) the angle **H**,
(*d*) the length of **HD**,
(*e*) the angle **K**,
(*f*) the length of **DK**,
(*g*) the angle **G**.

20. The horizontal distance that a projectile travels is given by the formula $d = v^2 \dfrac{\sin A \cos A}{16}$. In this formula d is the horizontal distance in feet,

v is the velocity at which it travels in feet per second and A is the angle of elevation at which it is projected. Find the horizontal distance that a body travels if its starting velocity is 30 feet per second and the angle of elevation is

(*a*) 30°, (*b*) 40°, (*c*) 45°, (*d*) 50°, (*e*) 60°, (*f*) 70°.

21. **A**, **B**, **C**, **D** and **E** are the centres of 5 equally spaced holes on the circumference of a circle of radius 4″. Calculate the distance from centre **A** to the centre of each of the other holes.

TABLE OF TRIGONOMETRIC RATIOS

Angle	Sin	Cos	Tan	Angle	Sin	Cos	Tan
1°	·0175	·9998	·0175	46°	·7193	·6947	1·0355
2°	·0349	·9994	·0349	47°	·7314	·6820	1·0724
3°	·0523	·9986	·0524	48°	·7431	·6691	1·1106
4°	·0698	·9976	·0699	49°	·7547	·6561	1·1504
5°	·0872	·9962	·0875	50°	·7660	·6428	1·1918
6°	·1045	·9945	·1051	51°	·7771	·6293	1·2349
7°	·1219	·9925	·1228	52°	·7880	·6157	1·2799
8°	·1392	·9903	·1405	53°	·7986	·6018	1·3270
9°	·1564	·9877	·1584	54°	·8090	·5878	1·3764
10°	·1736	·9848	·1763	55°	·8192	·5736	1·4281
11°	·1908	·9816	·1944	56°	·8290	·5592	1·4826
12°	·2079	·9781	·2126	57°	·8387	·5446	1·5399
13°	·2250	·9744	·2309	58°	·8480	·5299	1·6003
14°	·2419	·9703	·2493	59°	·8572	·5150	1·6643
15°	·2588	·9659	·2679	60°	·8660	·5000	1·7321
16°	·2756	·9613	·2867	61°	·8746	·4848	1·8040
17°	·2924	·9563	·3057	62°	·8829	·4695	1·8807
18°	·3090	·9511	·3249	63°	·8910	·4540	1·9626
19°	·3256	·9455	·3443	64°	·8988	·4384	2·0503
20°	·3420	·9397	·3640	65°	·9063	·4226	2·1445
21°	·3584	·9336	·3839	66°	·9135	·4067	2·2460
22°	·3746	·9272	·4040	67°	·9205	·3907	2·3559
23°	·3907	·9205	·4245	68°	·9272	·3746	2·4751
24°	·4067	·9135	·4452	69°	·9336	·3584	2·6051
25°	·4226	·9063	·4663	70°	·9397	·3420	2·7475
26°	·4384	·8988	·4877	71°	·9455	·3256	2·9042
27°	·4540	·8910	·5095	72°	·9511	·3090	3·0777
28°	·4695	·8829	·5317	73°	·9563	·2924	3·2709
29°	·4848	·8746	·5543	74°	·9613	·2756	3·4874
30°	·5000	·8660	·5774	75°	·9659	·2588	3·7321
31°	·5150	·8572	·6009	76°	·9703	·2419	4·0108
32°	·5299	·8480	·6249	77°	·9744	·2250	4·3315
33°	·5446	·8387	·6494	78°	·9781	·2079	4·7046
34°	·5592	·8290	·6745	79°	·9816	·1908	5·1446
35°	·5736	·8192	·7002	80°	·9848	·1736	5·6713
36°	·5878	·8090	·7265	81°	·9877	·1564	6·3138
37°	·6018	·7986	·7536	82°	·9903	·1392	7·1154
38°	·6157	·7880	·7813	83°	·9925	·1219	8·1443
39°	·6293	·7771	·8098	84°	·9945	·1045	9·5144
40°	·6428	·7660	·8391	85°	·9962	·0872	11·430
41°	·6561	·7547	·8693	86°	·9976	·0698	14·301
42°	·6691	·7431	·9004	87°	·9986	·0523	19·081
43°	·6820	·7314	·9325	88°	·9994	·0349	28·636
44°	·6947	·7193	·9657	89°	·9998	·0175	57·290
45°	·7071	·7071	1·0000				

TABLES

English System of Units of Measurement

Measurement of Distance

Linear Units

1,000 mils = 1 in.
12 in. = 1 ft.
3 ft. = 1 yd.
$5\frac{1}{2}$ yd. = 1 rd.
320 rd. = 1 mi. (statute).

Printer's Units

1 in. = 72 points.
6 points = 1 nonpareil.
12 points = 1 pica.
1 point = ·013833 in.

Surveyor's Units

7·92 in. = 1 link.
100 links = 1 chain.
80 chains = 1 mi.
1 Gunter's ch. = 66 ft.
1 Engineer's ch. = 100 ft.

Nautical Units

1 fathom = 6 ft.
1 nautical mi. = 6,080 ft.
1 n. mi. = $\frac{8}{33}$ s. mi.
1 knot is a sailing rate of 1 n. mi. per hr.

Measurement of Area

144 sq. in. = 1 sq. ft.
9 sq. ft. = 1 sq. yd.
$30\frac{1}{4}$ sq. yd. = 1 sq. rd.
160 sq. rd. = 1 ac.
640 ac. = 1 sq. mi.
10 sq. ch. = 1 ac.

Measurement of Volume

1,728 cu. in. = 1 cu. ft. 27 cu. ft. = 1 cu. yd.
1 cord is the amount of wood in a pile 8 ft. long, 4 ft. wide and 4 ft. deep.
1 board foot is the amount of lumber in a board 1 ft. long, 1 ft. wide and 1 in. thick.

Measurement of Capacity

Units of Liquid Measurement

2 cups = 1 pt. (American)
2 pt. = 1 qt.
4 qt. = 1 gal.

Units of Dry Measurement

2 pt. = 1 qt.
4 qt. = 1 gal.
2 gal. = 1 pk.
4 pk. = 1 bu.

3 teaspoonfuls = 1 tablespoonful (American).
2 tablespoonfuls = 1 fluid ounce.
16 American fluid ounces = 1 pint (American).
1 American fluid ounce = 1·04 Imperial fluid ounces (approximately).
20 Imperial fluid ounces = 1 pint (Imperial).

Measurement of Capacity and Volume

1 Imp. gal. = 277·420 cu. in.
1 cu. ft. = 6·232 Imp. gal.
1 Am. gal. = 231 cu. in.
1 cu. ft. = 7·48 Am. gal.
1 Imp. bu. = 2219·36 cu. in.
1 cu. ft. = 0·78 Imp. bu.
1 Am. bu. = 2150·42 cu. in.
1 cu. ft. = 0·81 Am. bu.

Measurement of Weight

1 lb. Avoirdupois = 7,000 grains.
1 lb. Troy = 5,760 grains.
16 drams = 1 oz.
16 oz. = 1 lb. (Avoirdupois)

Canadian	British
100 lb. = 1 cwt.	14 lb. = 1 stone.
20 cwt. = 1 ton.	8 stone = 1 cwt.
1 short ton = 2,000 lb.	20 cwt. = 1 long ton.
	1 long ton = 2,240 lb.

1 Imp. gal. of water weighs 10 lb.
1 Am. gal. of water weighs 8·3267 lb.

Miscellaneous Units of Measurement

Paper	Small Articles
24 sheets = 1 quire.	12 units = 1 doz.
20 quires = 1 ream.	12 doz. = 1 gross.
	20 gross = 1 great gross.

Metric System of Units of Measurement

Measurement of Distance	Measurement of Area
10 mm. = 1 cm.	100 sq. mm. = 1 sq. cm.
10 cm. = 1 dm.	100 sq. cm. = 1 sq. dm.
10 dm. = 1 m.	100 sq. dm. = 1 sq. m.
10 m. = 1 Dm.	100 sq. m. = 1 sq. Dm.
10 Dm. = 1 Hm.	100 sq. Dm. = 1 sq. Hm.
10 Hm. = 1 Km.	100 sq. Hm. = 1 sq. Km.
1 m. = 100 cm.	1 hectare = 10,000 sq. m.
1 m. = 1,000 mm.	

Measurement of Volume	Measurement of Capacity
1,000 c.mm. = 1 c.c.	10 ml. = 1 cl.
1,000 c.c. = 1 c.dm.	10 cl. = 1 dl.
1,000 c.dm. = 1 c.m.	10 dl. = 1 l.
	10 l. = 1 Dl.
	10 Dl. = 1 Hl.
	10 Hl. = 1 Kl.

1 c.dm. = 1 l.
1 c.c. = 1 ml.

Measurement of Weight

10 mg. = 1 cg.
10 cg. = 1 dg.
10 dg. = 1 g.
10 g. = 1 Dg.
10 Dg. = 1 Hg.
10 Hg. = 1 Kg.
1 metric ton = 1,000 Kg.
1 gram is the weight of 1 c.c. of water.

Comparison of the English and the Metric Units of Measurement

To convert from the English to the Metric system	To convert from the Metric to the English system
Measurement of Distance	
1 in. = 2·54 cm.	1 cm. = 0·3937 in.
1 mi. = 1·61 Km.	1 Km. = 0·621 mi.
Measurement of Area	
1 sq. in. = 6·4516 sq. cm.	1 sq. cm. = 0·155 sq. in.
1 sq. yd. = 0·8361 sq. m.	1 sq. m. = 1·196 sq. yd.
1 ac. = 0·4047 hectare.	1 hectare = 2·47 ac.
Measurement of Volume	
1 cu. in. = 16·387064 c.c.	1 c.c. = 0·06102 cu. in.
1 cu. ft. = 0·02831 c.m.	1 c.m. = 35·3163 cu. ft.
Measurement of Capacity	
1 Imp. gal. = 4·54 l.	1 l. = 0·22 Imp. gal.
Measurement of Weight	
1 oz. = 28·35 g.	1 g. = 0·03527 oz.
1 lb. = 0·5436 Kg.	1 Kg. = 2·2046 lb.
1 Canadian ton = 907·2 Kg.	1 metric ton = 2204·6 lb.

DECIMAL EQUIVALENTS

$\frac{1}{64}$ ·015625	$\frac{11}{32}$ — ·34375	$\frac{43}{64}$ ·671875
$\frac{1}{32}$ — ·03125	$\frac{23}{64}$ ·359375	$\frac{11}{16}$ —— ·6875
$\frac{3}{64}$ ·046875	$\frac{3}{8}$ —— ·375	$\frac{45}{64}$ ·703125
$\frac{1}{16}$ —— ·0625	$\frac{25}{64}$ ·390625	$\frac{23}{32}$ — ·71875
$\frac{5}{64}$ ·078125	$\frac{13}{32}$ — ·40625	$\frac{47}{64}$ ·734375
$\frac{3}{32}$ — ·09375	$\frac{27}{64}$ ·421875	$\frac{3}{4}$ —— ·750
$\frac{7}{64}$ ·109375	$\frac{7}{16}$ —— ·4375	$\frac{49}{64}$ ·765625
$\frac{1}{8}$ —— ·125	$\frac{29}{64}$ ·453125	$\frac{25}{32}$ — ·78125
$\frac{9}{64}$ ·140625	$\frac{15}{32}$ — ·46875	$\frac{51}{64}$ ·796875
$\frac{5}{32}$ — ·15625	$\frac{31}{64}$ ·484375	$\frac{13}{16}$ —— ·8125
$\frac{11}{64}$ ·171875	$\frac{1}{2}$ —— ·500	$\frac{53}{64}$ ·828125
$\frac{3}{16}$ —— ·1875	$\frac{33}{64}$ ·515625	$\frac{27}{32}$ — ·84375
$\frac{13}{64}$ ·203125	$\frac{17}{32}$ — ·53125	$\frac{55}{64}$ ·859375
$\frac{7}{32}$ — ·21875	$\frac{35}{64}$ ·546875	$\frac{7}{8}$ —— ·875
$\frac{15}{64}$ ·234375	$\frac{9}{16}$ —— ·5625	$\frac{57}{64}$ ·890625
$\frac{1}{4}$ —— ·250	$\frac{37}{64}$ ·578125	$\frac{29}{32}$ — ·90625
$\frac{17}{64}$ ·265625	$\frac{19}{32}$ — ·59375	$\frac{59}{64}$ ·921875
$\frac{9}{32}$ — ·28125	$\frac{39}{64}$ ·609375	$\frac{15}{16}$ —— ·9375
$\frac{19}{64}$ ·296875	$\frac{5}{8}$ —— ·625	$\frac{61}{64}$ ·953125
$\frac{5}{16}$ —— ·3125	$\frac{41}{64}$ ·640625	$\frac{31}{32}$ — ·96875
$\frac{21}{64}$ ·328125	$\frac{21}{32}$ — ·65625	$\frac{63}{64}$ ·984375

ANSWERS

Page 6

12. (*a*) 20·32. (*b*) 22·86. (*c*) 38·10. (*d*) 60·96. (*e*) 91·44. **13.** (*a*) 3·15. (*b*) 3·94. (*c*) 39·37. (*d*) 1·18.

Page 11

7. (i) 45. (ii) 90. (iii) 45. (iv) 112·5. **8.** (*a*) 135. (*b*) 180. (*c*) 270. (*d*) 315. (*e*) 337·5. (*f*) 157·5.

Page 12

9. (*a*) N.E. (*b*) E. (*c*) S.E. (*d*) S.S.E. (*e*) S.S.W. (*f*) W.N.W. **10.** (*a*) 40. (*b*) 155. (*c*) 75. **11.** (*a*) 240° 45′ 47″. (*b*) 565° 57′ 5″. **12.** (*a*) 43° 13′ 5″. (*b*) 129° 5′ 25″. (*c*) 22° 54′ 42″. **13.** 1200. **14.** (*a*) 13° 20′ 15″. (*b*) 26° 24′ 39″. (*c*) 92° 48′ 48″. **15.** 62° 6′ 52″. **16.** (*a*) 9° 30′. (*b*) 45° 47′ 30″.

Page 16

15. $5\frac{7}{8}$. **16.** $6\frac{1}{8}$. **17.** 16° 35′. **18.** (*a*) 56° 18′; 131° 43′; 227° 28′. (*b*) 132° 32′.

Pages 17 and 18

Review Exercise

14. (*a*) 20·32. (*b*) 30·48. (*c*) 76·20. (*d*) 91·44. (*e*) 100. **15.** (*a*) 3·937. (*b*) 39·37. (*c*) 0·3937. (*d*) 1·969. (*e*) 0·03937. (*f*) 0·5906. **16.** 184. **17.** (*a*) 14,400. (*b*) 72,000. (*c*) 60. **18.** (i) $22\frac{1}{2}$. (ii) E. 26° S. **19.** (*a*) 141° 11′ 52″. (*b*) 35° 17′ 58″. **20.** (*a*) 49° 27′ 30″. (*b*) 23° 54′ 45″.

Pages 22 and 23

1. $59\frac{1}{2}°$, 68°, $52\frac{1}{2}°$. **2.** 113°, $33\frac{1}{2}°$, $33\frac{1}{2}°$. **3.** 37°, 90°, 53°. **4.** 60°, 60°, 60°.

Page 26

6. $3\frac{1}{8}$, $53\frac{1}{2}°$, $69\frac{1}{2}°$. **7.** 5, $37\frac{1}{2}°$, $32\frac{1}{2}°$. **8.** $3\frac{1}{8}$, $3\frac{3}{4}$, 60°. **9.** $12\frac{7}{16}$, 70°. **10.** $2\frac{1}{2}$, $2\frac{3}{8}$, 55°. **12.** (*a*) 41°, 83°, 56°. (*b*) $14\frac{3}{4}$, $39\frac{1}{2}°$, $30\frac{1}{2}°$. (*c*) $50\frac{3}{8}$, $42\frac{3}{4}$, 55°.

Page 29

8. $1\frac{1}{4}$. **9.** $2\frac{1}{4}$.

Page 33

6. $1\frac{3}{4}$.

Page 36

9. 5·8.

Page 37

Review Exercise

10. $59\frac{1}{2}°$. **11.** $2\frac{3}{8}$. **12.** $1\frac{5}{8}$.

Pages 40 and 41

4. 14. **5.** 30. **6.** 54. **7.** 8,152¼. **8.** (*a*) $5\frac{3}{8}$. (*b*) 5·375. **9.** (*a*) 1·500. (*b*) ¾; 1½; $3\frac{3}{8}$; $5\frac{5}{8}$. **10.** (*a*) $3\frac{1}{4}$. (*b*) $\frac{21}{32}$. (*c*) $\frac{7}{16}$. **11.** (*a*) $11\frac{1}{2}$. (*b*) $8\frac{1}{8}$. **12.** (*a*) 4. (*b*) 12. (*c*) 8. (*d*) $4\frac{4}{5}$. **13.** 4·90. **14.** $4\frac{1}{4}$; $37\frac{11}{17}$. **15.** 85·75.

Pages 43 and 44

19. 30. **20.** 5·5. **21.** 2·0. **22.** 42. **23.** 3. **24.** 1,875. **25.** $4\frac{8}{15}$. **26.** $\frac{2}{5}$. **27.** 3. **28.** 39. **29.** 1·85. **30.** 0·04515. **31.** 0·425.

Pages 44, 45 and 46

7. 2,763. **9.** 312,885. **10.** 2,240. **11.** (*a*) 13. (*b*) 5. **12.** $14\frac{2}{5}$. **13.** 289. **14.** 1,331. **15.** 1,108. **16.** (*a*) 353′ 6″. (*b*) 146′ 6″. **17.** 20. **18.** (*a*) 16. (*b*) 3. **19.** $7\frac{1}{9}$. **20.** (*a*) 23. (*b*) 31. (*c*) 13. (*d*) 9. (*e*) 45; 21. **21.** 510.

Page 47

9. (*a*) 1. (*b*) 54. (*c*) 33. (*d*) 81. (*e*) 81. (*f*) 401. (*g*) 45. **10.** (*a*) 850. (*b*) 320. (*c*) 7,817. (*d*) 95. (*e*) 289. (*f*) 195.

Page 49

8. (*a*) $2^2 \times 19$. (*b*) 2×3^3. (*c*) 7×13. (*d*) 7^3. (*e*) $2^4 \times 3$. (*f*) $2 \times 5 \times 11$. (*g*) $2^4 \times 3^2$. (*h*) 2^9. **9.** (*a*) $3^4 \times 5^2$. (*b*) $3^2 \times 5^4$. (*c*) $3^2 \times 7^2$. **10.** (*a*) $2 \times 3 \times 5 \times 7 \times 11$. (*b*) $2^2 \times 3^2 \times 5 \times 7$. (*c*) $3^2 \times 5 \times 11$. (*d*) $2 \times 3 \times 7 \times 13 \times 17$. (*e*) $2^4 \times 3^4$. **11.** (*a*) 5, 45; 3, 75; 15, 15; 9, 25. (*b*) 100, 156, 60, 68. (*c*) the square.

Page 51

5. (*a*) 11. (*b*) 14. **6.** (*a*) 90. (*b*) 120. (*c*) 40. (*d*) 15. (*e*) 25. (*f*) 20. (*g*) 18. (*h*) 1·5. **7.** (*a*) 45. (*b*) 35. (*c*) 21. (*d*) 51. (*e*) 56. (*f*) 88. (*g*) 39. (*h*) 105. **8.** 63. **9.** 35. **10.** 60. **11.** (*a*) 5. (*b*) 6. (*c*) 15. (*d*) 7. (*e*) 9. (*f*) 11. **12.** 40.

Page 53

6. (*a*) 144. (*b*) 336. (*c*) 455. (*d*) 510. (*e*) 570. (*f*) 264. (*g*) 504. (*h*) 2,760. **7.** (*a*) 2,530. (*b*) 30,030. (*c*) 31,185. (*d*) 21,645. **8.** 120.

Pages 55 and 56

Review Exercise

12. (*a*) 8. (*b*) 12. (*c*) 72. (*d*) 120. (*e*) 200. (*f*) 72. **13.** 15. **14.** 60. **15.** 21; 6. **16.** 766. **17.** 28. **18.** 27. **20.** 1,043. **21.** (*a*) $2 \times 3^2 \times 5$. (*b*) 2×7^2. (*c*) $2^4 \times 7$. (*d*) 2^7. (*e*) $2^2 \times 3 \times 13$. (*f*) $5^2 \times 7^2$. (*g*) 3^6.

(*h*) $2^2 \times 3^2 \times 5 \times 7$. **22.** (*a*) 15. (*b*) 18. (*c*) 22. (*d*) 26. (*e*) 28. (*f*) 39. (*g*) 54. (*h*) 84. (*i*) 110. **23.** (*a*) 2^6. (*b*) 10^3. (*c*) 2^8. (*d*) 12^3. (*e*) 15^3. **24.** (*a*) 1,125. (*b*) 324. (*c*) 600. (*d*) 210. **25.** (*a*) 18. (*b*) 6. (*c*) 27. (*d*) 794. (*e*) 2. (*f*) 69.

Pages 59 and 60

12. (*a*) $\frac{15}{17}$. (*b*) $\frac{3}{5}$. (*c*) $\frac{23}{25}$. (*d*) $\frac{9}{16}$. (*e*) $\frac{1}{4}$. (*f*) $\frac{1}{11}$. (*g*) $\frac{1}{24}$. (*h*) $\frac{101}{225}$. (*i*) $\frac{2}{99}$. **13.** (*a*) 42. (*b*) $98\frac{7}{9}$. (*c*) 49. (*d*) 231. (*e*) 325. (*f*) 456. (*g*) 32. (*h*) 75. (*i*) $6\frac{2}{3}$. **14.** (*a*) 75. (*b*) $12\frac{1}{2}$. (*c*) $37\frac{1}{2}$. (*d*) $62\frac{1}{2}$. (*e*) $33\frac{1}{3}$. (*f*) $14\frac{2}{7}$. (*g*) $\frac{1}{2}$. (*h*) $\frac{1}{3}$. (*i*) $\frac{1}{8}$. (*j*) $1\frac{1}{2}$. (*k*) $2\frac{1}{3}$.

Pages 62 and 63

5. $1\frac{59}{64}$. **6.** $3\frac{3}{32}$. **7.** $3\frac{1}{64}$. **8.** (*a*) 1,575. (*b*) 756. (*c*) 510. (*d*) 216. **9.** (*a*) $\frac{13}{16}$. (*b*) $1\frac{17}{32}$. (*c*) $2\frac{3}{64}$. (*d*) $2\frac{21}{64}$. (*e*) $2\frac{3}{32}$. (*f*) $2\frac{33}{64}$. **10.** (*a*) $2\frac{1}{4}$. (*b*) $\frac{9}{16}$. (*c*) $\frac{5}{8}$. (*d*) $1\frac{1}{20}$. **11.** (*a*) $\frac{83}{240}$. (*b*) $\frac{1}{3}$. (*c*) $\frac{1}{4}$. (*d*) $\frac{161}{216}$. (*e*) $\frac{1427}{1575}$. (*f*) $\frac{409}{756}$. **12.** (*a*) $2\frac{31}{64}$. (*b*) $\frac{11}{32}$. **13.** (*a*) $\frac{13}{32}$. (*b*) $\frac{9}{64}$. (*c*) $\frac{5}{8}$. (*d*) $1\frac{3}{10}$. **14.** (*a*) $1\frac{143}{144}$. (*b*) $1\frac{57}{120}$. (*c*) $\frac{19}{132}$. (*d*) $\frac{89}{240}$. **15.** $3\frac{55}{64}$; $1\frac{13}{64}$; $\frac{59}{64}$. **16.** $\frac{3}{64}$. **17.** (*a*) $\frac{9}{16}$, $\frac{17}{32}$, $\frac{1}{2}$, $\frac{31}{64}$. (*c*) $\frac{5}{6}$, $\frac{3}{4}$, $\frac{7}{10}$, $\frac{2}{3}$, $\frac{7}{15}$, $\frac{5}{12}$. **18.** $1\frac{9}{32}$; $3\frac{29}{32}$; $\frac{31}{32}$; $\frac{3}{4}$.

Pages 65, 66 and 67

6. $\frac{7}{8}$. **7.** $6\frac{7}{16}$. **8.** $6\frac{7}{8}$. **9.** $7\frac{13}{16}$. **10.** $13\frac{17}{90}$. **11.** $8\frac{5}{24}$. **12.** $3\frac{5}{8}$. **13.** $298\frac{13}{16}$. **14.** $229\frac{17}{40}$. **15.** $\frac{5}{8}$; $\frac{9}{16}$; $\frac{3}{4}$. **16.** $12\frac{1}{2}$. **17.** $8\frac{9}{16}$. **18.** $\frac{11}{16}$. **19.** $9\frac{3}{4}$; $7\frac{1}{4}$. **20.** $14\frac{1}{16}$. **21.** $15\frac{7}{15}$. **22.** $10\frac{3}{5}$. **23.** $11\frac{3}{32}$. **24.** $10\frac{11}{32}$.

Pages 69 and 70

6. (*a*) 19. (*b*) $\frac{3}{8}$. (*c*) $11\frac{11}{14}$. (*d*) $38\frac{1}{2}$. (*e*) $15\frac{1}{2}$. (*f*) $30\frac{1}{4}$. (*g*) $1\frac{2}{3}$. (*h*) $5\frac{2}{5}$. (*i*) $2\frac{1}{2}$. (*j*) $21\frac{13}{24}$. **7.** (*a*) 55. (*b*) $36\frac{1}{9}$. (*c*) $104\frac{1}{6}$. **8.** 55; $4\frac{1}{8}$; $24\frac{3}{4}$. **9.** (*a*) $7\frac{6}{7}$. (*b*) 11. (*c*) $16\frac{1}{2}$. **10.** $52\frac{1}{2}$. **11.** (*b*) $14\frac{1}{4}$. **12.** (*a*) $1\frac{1}{8}$; $1\frac{3}{8}$; $1\frac{1}{8}$. (*b*) $3\frac{5}{8}$. **13.** (*a*) $11\frac{3}{8}$. (*b*) $22\frac{3}{4}$ or $25\frac{5}{8}$ or $31\frac{1}{4}$. **14.** $\frac{343}{512}$. **15.** $20\frac{7}{8}$. **16.** 9. **17.** 33·47.

Pages 72 and 73

3. (*a*) $5\frac{1}{3}$. (*b*) $\frac{3}{22}$. (*c*) $\frac{1}{2}$. (*d*) $1\frac{1}{8}$. (*e*) $\frac{1}{6}$. (*f*) $\frac{2}{5}$. (*g*) $2\frac{1}{2}$. (*h*) 10. (*i*) $\frac{1}{10}$. **4.** (*a*) 35. (*b*) 9. (*c*) $\frac{1}{10}$. (*d*) $\frac{3}{200}$. (*e*) $\frac{3}{400}$. (*f*) $266\frac{2}{3}$. **5.** 8. **6.** $\frac{1}{32}$. **7.** 192. **8.** 143. **9.** 436; 1. **10.** (*a*) $1\frac{1}{64}$. (*b*) $\frac{11}{32}$. **11.** $\frac{29}{96}$.

Pages 74 and 75

2. $3\frac{15}{17}$. **3.** 224. **4.** (*a*) 5. (*b*) $\frac{1387}{}$. **5.** (*a*) $1\frac{1683}{}$. (*b*) $\frac{55}{96}$. (*c*) $42\frac{2}{3}$. (*d*) $\frac{13}{30}$. **6.** (*a*) $2\frac{2}{3}$. (*b*) $25\frac{7}{16}$. (*c*) $1\frac{22}{45}$. (*d*) $6\frac{3}{32}$. **7.** (*a*) 12. (*b*) 3. (*c*) $\frac{46}{65}$. **8.** (*a*) $1\frac{3}{5}$. (*b*) $1\frac{7}{10}$. **9.** $\frac{5}{36}$. **10.** (*a*) $\frac{1955}{}$. (*b*) $\frac{3}{10}$. (*c*) $1\frac{3}{7}$. (*d*) $3\frac{1}{3}$. (*e*) $1\frac{1}{44}$. (*f*) $1\frac{15}{16}$. (*g*) $2\frac{13}{20}$. (*h*) $\frac{36}{115}$.

Pages 78 and 79

4. 7·75. **5.** 0·35. **6.** (*a*) 8·450. (*b*) 1·193. (*c*) 0·789. (*d*) 0·700. **7.** 0·277. **8.** 61·127. **9.** (*a*) 0·613. (*b*) 0·965. **10.** 15·50. **12.** 3. **13.** (i) 14·6; 18·3; 7·2; 15.4. (ii) 14·6; 32·9; 40·1. (iii) 55·5. (iv) 14·6; 22·6; 15·4. **14.** (*a*) 23·34, −1·66; 26·66, +1·66; 24·43, −0·57; 20·74, −4·26. (*b*) 95·17, −4·83.

Page 84

17. (*a*) 30·28. (*b*) 114′ 2·5″. (*c*) 17 yd. 2 ft. 7 in. **18.** (*a*) 77·95. (*b*) 28 lb. 7 oz. (*c*) 84·4. **19.** (*a*) 7′ 7″. (*b*) 4 yd. (*c*) 229.

Pages 85 and 86

5. (*a*) 52·89. (*b*) 10,395·2. (*c*) 5·24552. (*d*) 31·36. (*e*) 0·571787. (*f*) 768·553. **6.** 8·975. **7.** 2·515456. **8.** 389. **10.** (*a*) 2·188. (*b*) 3·125. (*c*) 3·438. **11.** 112·5. **12.** 5·751. **13.** 0·463, 0·106; 0·694, 0·159; 0·348, 0·080; 0·231, 0·053. **14.** 50,211. **15.** 176·7. **16.** 5·96.

Pages 88 and 89

4. (*b*) 0·34875. (*d*) 0·38775. **5.** 21·5. **6.** (*a*) 0·0893. (*b*) 0·223. (*c*) 0·0632.

Pages 92 and 93

6. 0·014. **7.** 167. **8.** 0·3250. **9.** 0·095. **10.** 0·0045. **11.** 5,714. **12.** 0·002. **13.** 34·9. **14.** (*a*) 0·88. (*b*) 0·0785. **15.** (*a*) 40. (*b*) 16. (*c*) 51. **16.** 1,496. **17.** (*a*) 6·65. (*b*) 0·504.

Pages 94 and 95

3. (*a*) 0·143; 0·286; 0·429; 0·571; 0·714; 0·857. (*b*) 0·111; 0·222; 0·333; 0·444; 0·556; 0·667; 0·778; 0·889. (*c*) 0·0909; 0·182; 0·273; 0·364; 0·455; 0·545; 0·636; 0·727; 0·818; 0·909. (*d*) 0·0833; 0·167; 0·250; 0·333; 0·417; 0·500; 0·583; 0·667; 0·750; 0·833; 0·917. **6.** (*a*) $0\dot{3}$. (*b*) $0\dot{6}$. (*c*) $0{\cdot}8\dot{3}$. (*d*) $0{\cdot}\dot{2}$. (*e*) $0{\cdot}\dot{0}\dot{9}$. **7.** (*a*) $\frac{1}{8}$. (*b*) $\frac{7}{40}$. (*c*) $\frac{33}{100}$. (*d*) $\frac{1}{40}$. (*e*) $\frac{1}{160}$. (*f*) $\frac{7}{160}$. (*g*) $\frac{3}{32}$. (*h*) $\frac{13}{160}$. (*i*) $\frac{39}{40}$. **8.** (*a*) $\frac{11}{32}$. (*b*) $\frac{11}{16}$. (*c*) $\frac{13}{16}$. (*d*) $\frac{19}{64}$. (*e*) $\frac{11}{64}$. (*f*) $\frac{3}{64}$. (*g*) $\frac{33}{64}$. (*h*) 1. **9.** (*a*) $\frac{3}{4}$. (*b*) $\frac{7}{16}$. (*c*) $3\frac{1}{7}$. (*d*) 0·1895. (*e*) 0·438. (*f*) $\frac{17}{32}$. **12.** (*a*) $\frac{3}{14}, \frac{7}{40}, \frac{5}{32}$. (*b*) $\frac{15}{21}, \frac{7}{10}, \frac{9}{15}$. (*c*) $\frac{7}{10}, \frac{9}{13}, \frac{15}{24}$. (*d*) $2\frac{59}{100}$, $2\frac{7}{12}$, $2\frac{8}{15}$.

Pages 96, 97, and 98

Review Exercise

3. 437·5. **4.** 26. **5.** (*a*) $9\frac{1}{4}$. (*b*) 8·875. **6.** 8·08. **7.** 9·72; 22·7. **8.** (*a*) 2·17. (*b*) 6·1. (*c*) 1·3. **9.** (*a*) 0·0812. (*b*) 0·0406. (*c*) 0·0325. **10.** 118·75. **11.** $\frac{1}{14}$. **12.** (*a*) 151. (*b*) Sept. 7. **13.** (*a*) 16·88. (*b*) 14·63. (*c*) 22·13. (*d*) 10·28. **14.** 0·44, 0·19; 0·33, 0·14; 0·27, 0·12. **15.** (*a*) 8·1. (*b*) 11·0.

Pages 100 and 101

11. 20; $6\frac{2}{3}$; $3\frac{1}{3}$. **12.** 25; 20; $8\frac{1}{3}$. **13.** 20; 200. **14.** 15. **15.** 10. **16.** 70. **17.** $9\frac{3}{13}$. **18.** $15\frac{15}{19}$. **19.** $76\frac{2}{3}$. **20.** (*a*) $62\frac{1}{2}$. (*b*) $128\frac{4}{7}$. (*c*) $5\frac{5}{9}$. **21.** (*a*) $57\frac{1}{7}$. (*b*) $87\frac{1}{2}$. (*c*) $66\frac{2}{3}$. **22.** (*a*) $38\frac{8}{9}$. (*b*) 150. (*c*) 400. (*d*) $53\frac{4}{7}$. **23.** 56·3; 13·9; 2·0; 22·6; 5·2. **24.** 25. **25.** 4.

Page 103

5. 1·20. **6.** 6·16. **7.** 16·50, 14·85. **8.** 212·5; 25; 12·5. **9.** 205. **10.** 40; 75; 385. **11.** 16,300. **12.** 2,637,566,000.

Pages 104 and 105

1. 216·00. **2.** 175·50; 84·00; 286·00; 481·50; 1,355·80. **3.** 202·35; 199·31. **4.** (*a*) 2·5. (*b*) 4·50. **5.** 52. **6.** 21.

Pages 107 and 108

1. 175. **2.** (*a*) 300. (*b*) 270. **3.** 45·28. **4.** 150. **5.** 90. **6.** (*a*) 20; 6. (*b*) 62·50; 18·75. (*c*) 36; 10·80. **7.** 94. **8.** 32. **9.** $6\frac{1}{4}$.

Pages 108 and 109

1. 20; 80. **2.** 5; 4; 4·38. **3.** 280; 7,720. **4.** (*a*) 30. (*b*) 750. (*c*) 90. **5.** 455. **6.** 23·70. **7.** 1·13.

Pages 109 and 110

Review Exercise

1. 4,800; 6,400; 4,800. **2.** 8,000; 43,200; 17,280; 11,520. **3.** (*a*) $\frac{2}{25}$. (*b*) $\frac{7}{20}$. (*c*) $\frac{1}{200}$. (*d*) $2\frac{1}{4}$. (*e*) $\frac{13}{200}$. (*f*) $\frac{3}{200000}$. (*g*) $\frac{1}{80}$. (*h*) $\frac{1}{125}$. **4.** (*a*) 75. (*b*) $55\frac{5}{9}$. (*c*) 150. (*d*) 82·6. (*e*) 4·8. **5.** 1,523. **6.** 15. **7.** (*a*) 432. (*b*) 6·48. (*c*) 10·52. **8.** 268. **9.** (*a*) 2·21. (*b*) 41·25. **10.** 4·37. **11.** 3·61. **12.** 342·80. **13.** 900; 875; 850; 825; 800. **14.** (*a*) 74. (*b*) 1,530.

Pages 113, 114 and 115

6. 88. **7.** 242·25. **10.** (*a*) $25\frac{1}{4}$. (*b*) $4\frac{6}{11}$. (*c*) 192. (*d*) $35\frac{8}{9}$. (*e*) 3′ 2·4″. **11.** (*a*) 123. (*b*) $5\frac{2}{3}$. (*c*) 3,275. (*d*) $7\frac{2}{3}$. **12.** (*a*) 20′ 5″. (*b*) 7 yd. 1 in. (*c*) 1 mi. 180 rd. **13.** 7′ 6″. **14.** 2′ 10″. **15.** (*a*) 21′ 2″. (*b*) 30′ $10\frac{7}{8}$″. (*c*) 127′ $9\frac{25}{32}$″. **16.** 4′ 11″. **17.** 9′ 9″. **18.** 2′ $9\frac{3}{4}$″. **19.** (*a*) 52·8. (*b*) 39·6. (*c*) 35·2. (*d*) 26·4. (*e*) 17·6. **20.** (*a*) $\frac{38}{39}$. (*b*) 1·152. **21.** 23. **22.** (*a*) $\frac{33}{38}$. (*b*) 0·868. **23.** 26·0. **25.** 8. **26.** 99. **27.** 21; 15·5. **28.** (*a*) 3,024. (*b*) 11 ch. 25 l. (*c*) 2·18. (*d*) 490. (*e*) 0·05.

Pages 117, 118 and 119

8. 0·3937. **9.** 39·37. **10.** (*a*) 63·5. (*b*) 91·44. (*c*) 3·175. **11.** (*a*) 25·59. (*b*) 11·26. (*c*) 0·3504. **12.** (*a*) 56·4. (*b*) 12·1. (*c*) 547. **13.** (*a*) 14·9. (*b*) 11·6. (*c*) 2,012. **15.** (*a*) 1,068. (*b*) 980. (*c*) 3·055. (*d*) 5·376. **16.** (*a*) (i) 35·5. (ii) 355. (*b*) (i) 5·03. (ii) 503. (*c*) (i) 0·35. (ii) 35. (iii) 350. (*d*) (i) 5,002. (ii) 5·002. (*e*) (i) 7,003·05. (ii) 700,305. (iii) 7·00305. **17.** (*a*) 0·621. (*b*) 1·61. **18.** (*a*) 2 cm. 5 mm. (*b*) 4 dm. 5 cm. 3 mm. (*c*) 7 Km. 563 m. (*d*) 24 m. 5 dm. 3 cm. **19.** 50·617. **20.** 43·27 m. **22.** 750. **23.** 51·1. **24.** 72. **25.** 2·95. **26.** (*a*) 29·8. (*b*) 62·1. **27.** 10·9. **28.** 1·14.

Page 122

2. $25\frac{1}{2}$. **3.** 7. **4.** $p = 2x$. **5.** $44\frac{1}{2}$. **6.** $19\frac{3}{16}$. **7.** 88·3. **8.** 266·7. **9.** 443·9. **10.** 61·9. **11.** 71·0.

Pages 124, 125 and 126

6. 129. **7.** 22·0. **8.** 10·0. **9.** 25·1. **10.** (*a*) 89·2. (*b*) 91·1. (*c*) 94·2. **11.** 67·1. **12.** 8,290; 83. **13.** 6·4. **14.** (*a*) 3·2. (*b*) 15·9. (*c*) 8·0. (*d*) 3·2.

15. (*a*) 19·5. (*b*) 25·7. (*c*) 38·3. (*d*) 47·7. (*e*) 22·6. (*f*) 55·6. **16.** (*a*) 6·28. (*b*) 52·3.

Page 126

Review Exercise

1. 20. **2.** 30·80. **3.** 44·0. **4.** 1,360. **5.** 43. **6.** 1,095·6. **7.** 38·2. **8.** 660. **9.** 88. **10.** 1,170. **11.** 1,330. **12.** 8. **13.** 1·68. **14.** 115.

Pages 130 and 131

10. (*a*) $1\frac{5}{6}\frac{7}{4}$. (*b*) 2·2. **11.** (*a*) 14·4. (*b*) 9·7. **12.** (*a*) (i) 2·2. (ii) 13·8. (*b*) (i) $1\frac{2}{6}\frac{7}{4}$. (ii) 9·0. **13.** $8\frac{1}{8}$. **14.** 20. **16.** $61\frac{7}{8}$. **17.** 39. **18.** 1,350. **19.** 398. **20.** $379\frac{1}{2}$. **21.** 13·5. **22.** 91·52. **23.** 168. **24.** (*a*) 15. (*b*) 18. (*c*) 28. (*d*) 29.

Pages 133 and 134

1. 37. **2.** (*a*) 71. (*b*) 43. (*c*) 87. (*d*) 38. (*e*) 34. (*f*) 83. (*g*) 36. (*h*) 48. (*i*) 64. (*j*) 39. (*k*) 72. (*l*) 70. **3.** (*a*) 23. (*b*) 28. (*c*) 26. (*d*) 25. (*e*) 121. (*f*) 169. (*g*) 106. (*h*) 231. (*i*) 907. (*j*) 809. (*k*) 806. (*l*) 637. (*m*) 15,625. (*n*) 5,307. **4.** (*a*) 78. (*b*) 110. **5.** 165. **6.** (*a*) 65. (*b*) 100. **7.** (*a*) 200. (*b*) 55.

Page 135

3. 5·6. **4.** (*a*) 2·9. (*b*) 0·38. (*c*) 0·31. (*d*) 0·071. **5.** 3·6. **6.** (*a*) 31·6. (*b*) 0·737. (*c*) 5·20. **7.** (*a*) 1·414. (*b*) 1·732. (*c*) 2·236. **8.** (*a*) 3·12. (*b*) 1·56. (*c*) 5·2. **9.** 41·6.

Page 137

6. (*a*) 120. (*b*) 89. **7.** 179. **8.** 504. **9.** 51. **10.** 42.

Page 139

4. 60·1. **5.** 251. **6.** 880. **7.** 8. **8.** 75.

Pages 140, 141 and 142

8. 6·45. **9.** 4,840. **10.** 15. **11.** 20. **12.** 420. **13.** 79·2. **14.** 72. **15.** $2\frac{7}{6}\frac{7}{4}$. **16.** 33. **17.** 1·8. **18.** $89\frac{1}{4}$. **19.** 9·12. **20.** 49·92. **21.** 18·89. **22.** 576. **23.** 37·4. **24.** 4,084. **25.** 2,700. **26.** 529. **27.** 0·5. **28.** 5,354. **29.** (*a*) 72. (*b*) 5·9. (*c*) 1·4. **30.** 33. **31.** 107·25. **32.** 1,290.

Page 145

6. 26·6. **7.** 76·0. **8.** 15. **9.** (*a*) 16. (*b*) 32. **10.** 51·0. **11.** 21. **12.** $14\frac{1}{16}$.

Pages 147, 148 and 149

8. 96. **9.** 32·1. **10.** 20,000. **11.** $170\frac{1}{2}$. **12.** 3·5. **13.** 31·6. **14.** $1\frac{1}{16}\frac{1}{6}$; 2·5. **15.** (*a*) 90. (*b*) $1\frac{7}{8}$. (*c*) $1\frac{5}{3}\frac{6}{2}$. **16.** $48\frac{6}{6}\frac{3}{4}$. **17.** (*a*) xy. (*b*) $\frac{1}{2}az$. (*c*) $0{\cdot}435x^2$. (*d*) 323. (*e*) 215. (*f*) $1\frac{1}{4}x^2$. **18.** $17\frac{13}{16}$. **19.** 60.

Pages 152, 153 and 154

5. (*a*) 38·5. (*b*) 19·6. (*c*) 12,900. (*d*) 373. **6.** 616. **7.** 24·6. **8.** 628. **9.** 10·5. **10.** 308. **11.** 0·601. **12.** 2·5. **13.** 9. **14.** 192. **15.** 2·76. **16.** 113·4. **17.** 5·52. **18.** (*a*) 1·5; 9·42; 7·07. (*b*) .8; 25·1; 50·2. (*c*) 0·060; 0·377; 0·0113. (*d*) 0·70; 2·20; 0·385. **19.** (*a*) 10; 5; 78·5. (*b*) 74·8; 37·4; 4,390. (*c*) 14·0; 7·0; 154. **20.** (*a*) 10; 20; 62·8. (*b*) 5; 10; 31·4. (*c*) 6; 12; 37·7. **21.** 0·7854 : 1. **22.** 4 : 1. **23.** (*a*) 47·8. (*b*) 136. **24.** 36·4; 81·7. **25.** (*a*) 3. (*b*) 4·47. (*c*) 2·24. **26.** 2·1.

Page 158

3. (*a*) 20. (*b*) 5. (*c*) 8. **4.** 41. **5.** 24. **6.** No. **7.** 24. **8.** (*a*) 170. (*b*) 1,500. (*c*) 65. **9.** (*a*) 25. (*b*) 18·3. (*c*) 1·25. (*d*) 12·0. (*e*) 17·0. **10.** (*a*) 40. (*b*) 6·4. (*c*) 1·73. (*d*) 24·5. **11.** 5·6; 8·5.

Pages 160 and 161

Review Exercise

13. (*a*) $33\frac{1}{3}$. (*b*) 85. **14.** (*a*) 353. (*b*) 229. (*c*) 226. (*d*) 457. **15.** 250. **16.** 5. **17.** 8. **19.** 218. **20.** $2\frac{1}{8}$; $6\frac{3}{8}$. **21.** 140. **22.** (*a*) 35. (*b*) 72. (*c*) 12·0. **23.** 63·6. **24.** (*a*) 62. (*b*) 17. (*c*) 79. (*d*) 29. (*e*) 26. (*f*) 86. (*g*) 94. (*h*) 98. (*i*) 89. **25.** (*a*) 604. (*b*) 507. (*c*) 444. (*d*) 225. (*e*) 194. (*f*) 207. **26.** (*a*) 2·45. (*b*) 3·16. (*c*) 2·65. **27.** 336. **28.** 83·2. **29.** $56\frac{1}{4}$. **30.** 0·707. **31.** 10. **32.** 9.

Pages 164 and 165

9. 1,472. **10.** $95\frac{5}{8}$. **11.** 110. **12.** $293\frac{5}{8}$. **13.** 140. **14.** 72. **15.** $3\frac{3}{8}$. **16.** 4,480. **17.** $40\frac{5}{8}$. **18.** $85\frac{1}{3}$. **19.** $8\frac{1}{2}$. **20.** $\frac{15}{16}$. **21.** $67\frac{1}{2}$. **22.** 3·3. **23.** 7·7. **24.** (*a*) 240. (*b*) 8. **25.** (*a*) 25. (*b*) 5. **26.** 1′ 4″.

Pages 168 and 169

12. $4\frac{1}{2}$. **13.** 11·67. **14.** $97\frac{1}{2}$. **15.** 21·5. **16.** 126. **17.** $10\frac{2}{3}$. **18.** 84. **19.** 46·93. **20.** 5,760. **21.** 224. **22.** 105·84. **23.** 86·64. **24.** 3. **25.** 23·4. **26.** 30·38. **27.** $13\frac{1}{2}$.

Pages 171 and 172

8. $1{,}968\frac{3}{4}$. **9.** 37·38; 16·82. **10.** 40. **11.** 155·7. **13.** 420,000. **14.** 2,040. **15.** 1,496. **16.** 498·6. **17.** 797·7. **18.** 5,141. **19.** *b*.

Pages 174 and 175

4. 7,024; 7·024. **5.** 34·3; 264. **6.** 11,380,000. **7.** (*a*) 0·367. (*b*) 0·367. (*c*) 367. **8.** 0·52. **10.** (*a*) 2,204·6. (*b*) (i) 35·4. (ii) 204·6. **11.** (*a*) 17·3. (*b*) 5·20. (*c*) 13·8.

Pages 176 and 177

5. 32·8. **6.** 13,824. **7.** 16. **8.** (*a*) 20. (*b*) 18. (*c*) 21. (*d*) 25. (*e*) 24. **9.** 16. **10.** 45.

Pages 178, 179 and 180

7. $11\frac{1}{3}$. **8.** 15·64. **9.** 1,086. **10.** 25·2. **11.** 427·5. **12.** $21\frac{7}{8}$. **13.** 294. **14.** 5,520. **15.** $\frac{1}{2}$. **16.** (*a*) 56.

Pages 181 and 182

5. 115. **6.** 14·4. **7.** 1,960. **8.** 172. **9.** 1,840. **10.** 201. **11.** 6·15. **12.** 2,150. **13.** 275. **14.** 1,060. **15.** 73·6. **16.** (*a*) 2,261. (*b*) 490. **17.** 4·71; 61·2.

Pages 182, 183 and 184

Review Exercise

1. (*a*) $5\frac{2}{3}\frac{9}{2}$. (*b*) 1·7. **2.** (*a*) 18. (*b*) 450. **3.** 69·4; 34·7. **4.** 1·1. **5.** (*a*) 7. (*b*) 2,140. **6.** 90. **7.** 1,080. **8.** (*a*) 63·6. (*b*) 397. **9.** (*a*) 38·5. (*b*) 20·8. **10.** 509. **11.** 28·63. **12.** 6,230. **13.** 11,300. **14.** 195. **15.** 0·51. **16.** (*a*) 0·424. (*b*) 119.

Pages 191 and 192

11. (*a*) 26. (*b*) 0·9. (*c*) $1\frac{1}{2}$. (*d*) 33·94. **12.** (*a*) 10. (*b*) $\frac{2}{3}$. (*c*) 0·15. (*d*) $12\frac{3}{5}$. **13.** 16; 64; 144; 256; 400; 576. **14.** 56. **15.** 24. **16.** 5. **17.** 6. **18.** 113·04. **19.** 49. **20.** 255. **21.** (*a*) 16. (*b*) 21. (*c*) 45. (*d*) 33. (*e*) 15. (*f*) 54. **22.** (*a*) $\frac{1}{6}$. (*b*) $\frac{1}{6}$. (*c*) $1\frac{1}{2}$. (*d*) 2. (*e*) 5. (*f*) $\frac{1}{2}$. **23.** 25. **24.** $11\frac{2}{3}$. **25.** 3. **26.** 26. **27.** 20. **28.** 7. **29.** $24\frac{1}{2}$. **30.** 1,260, 0. **32.** $5\frac{5}{6}$. **33.** (*a*) 12. (*b*) 12. (*c*) 24. (*d*) 28. (*e*) 10. (*f*) 10. **34.** 20; 20; 44; 50; 22; 22.

Pages 194 and 195

11. (*a*) $9a^2 + a - 9$. (*b*) $2x + 5y + 5z$. (*c*) $9x + 5$. (*d*) $7ab + bc$. (*e*) $x^2 + 7x + 3$. (*f*) $6a + 6b$. **12.** 74. **13.** $\frac{3}{4}$. **14.** 10. **15.** (*a*) $2y^3 + y^2 + 3y + 1$. (*b*) $r^2 + 4r + 8$. (*c*) $6x^3 + 6x^2 - 6x + 3$. (*d*) $x^2 - 7x + 3$. (*e*) $1\frac{1}{4}x^2 + 1\frac{1}{8}x + \frac{9}{16}$. **16.** $7x$; x. **17.** $41a^2$. **18.** $52a^2$. **19.** (*a*) x^2. (*b*) hx. (*c*) $4hx + 2x^2$. **20.** (*a*) $4a - 5ab + 2b$. (*b*) $-12ab$. (*c*) $7a^2b + 2ab^2$. (*d*) $-bc + ab$. (*e*) $-x^2 + 12x^2y - 3xy^2$. (*f*) $4xy + 5x^2y + 4yz$. **21.** (*a*) 39. (*b*) $13x$. (*c*) $17b$. **22.** yes: no. **24.** 8; 3; 0; -1; 0; 3; 8. **25.** 2; 3. **26.** 2.

Pages 195 and 196

Review Exercise

7. (*a*) 14. (*b*) 18. (*c*) 50. (*d*) 36. (*e*) 12. (*f*) 9. **8.** (*a*) 40. (*b*) 192. (*c*) 4. (*d*) 256. (*e*) 98. (*f*) $5\frac{1}{3}$. **9.** (*a*) 43. (*b*) 72. (*c*) 24. (*d*) 36. (*e*) 17. (*f*) 90. (*g*) $1\frac{1}{12}$. (*h*) $2\frac{2}{11}$. (*i*) 24. (*j*) 15. **10.** (*a*) $4x^2 + x + 3$. (*b*) $7xy$. (*c*) $6a^2b - 3ab^2$.

Pages 204 and 205

19. $+6{\cdot}93$. **20.** $-8\frac{3}{8}$. **21.** $+1\frac{3}{8}$. **22.** $-\frac{39}{64}$. **23.** $-2\frac{4}{15}$. **24.** $2\frac{13}{18}$. **25.** $-16{\cdot}06$. **26.** $-0{\cdot}419$. **27.** $11a + 13b - 16c$. **28.** $10x - 14y + 14z$. **29.** $-11x^3 + 9x^2 + 13x$. **30.** $8ab + 13ac - 6bc$. **31.** $5x^2 - 15y - 10$. **32.** $4x^5 + 13x^4 - x^3$. **33.** $x^3 + 3x^2 + 3x + 1$. **34.** $36x$. **35.** $45x$. **36.** 44π. **37.** 15π.

38. $19x - 7y$. **39.** $-3x - 3y - 3z$. **40.** $9(a + b)$. **41.** 40. **42.** 83. **43.** $52\frac{1}{2}$. **44.** $14(x - y) + 11(y - z)$. **45.** $2(x - y)$.

Pages 206 and 207

17. $\frac{1}{4}$. **18.** $2\frac{3}{4}$. **19.** $1\frac{1}{4}$. **20.** $\frac{3}{32}$. **21.** $-0{\cdot}72$. **22.** $-0{\cdot}5$. **23.** (*a*) $+30°$. (*b*) $-15°$. (*c*) $+60°$. (*d*) $+270°$. **24.** $5a + 3b + 5c$. **25.** 24. **26.** $-a + b + c - 6$. **27.** $x^3 + 3x^2 + 15x - 18$. **28.** $4x^3 + x^2 - 50$. **29.** $4ab + 8b + 7c$. **30.** $6x + 5y$. **31.** 6. **32.** 1. **33.** $-2\frac{1}{2}$. **34.** $24n + 4$. **35.** $3n$. **36.** $5x + 10$. **37.** $8x - 12$. **38.** $10x + 5$. **39.** $3{\cdot}9a + 1{\cdot}3b + 3{\cdot}1c$. **40.** $-2{\cdot}88$. **41.** $2(a + b) - 2(a - c)$. **42.** $6(x + y) - 5(y - z)$.

Page 209

19. (*a*) $+3x$. (*b*) $+8a^2$. (*c*) $-5xy$. (*d*) $+3yz$. (*e*) $-7xz$. **20.** $4a + 4b$. **21.** $6a - b$. **22.** $-3a - 14b$. **23.** $-5x$. **24.** (*a*) $-x + 3b$. (*b*) $-2x^2 - 2x + 4$. (*c*) $2a - 4b - 12c$.

Pages 211 and 212

16. $a + 2b$. **17.** $2a - 3b$. **18.** $-2x + 6y$. **19.** a. **20.** $2y$. **21.** $-2a - 10b$. **22.** $-3x - 9y + 7z$. **23.** $9a + 7b + 2c$. **24.** $-2b + 3c$. **25.** $3a + 4b$. **26.** $3a - 3b$. **27.** $-b$. **28.** -4. **29.** $-x + y - 2$. **30.** $-a - b$. **31.** $y - 2$. **32.** $27y$. **33.** $-4x + 10$. **34.** $x - 18$. **35.** $-0{\cdot}6a + 1{\cdot}4b$. **36.** $0{\cdot}1a + 0{\cdot}6b$. **37.** $5a - 13b$. **38.** $-5a + b$. **41.** (*a*) $+8{\cdot}18$. (*b*) $-3{\cdot}87$. (*c*) $-0{\cdot}02$. (*d*) $-0{\cdot}27$. (*e*) $-0{\cdot}07$.

Pages 213 and 214

5. (*a*) $8x^2 - 6x - 10$. (*b*) $-a - 4b - 6c$. (*c*) $4x - y$. (*d*) $8a^2 - ab + 13b^2$. **6.** $8a + 6b$. **7.** (*a*) $-2x + 11y$. (*b*) $-2x^2 - 3x$. (*c*) $8a - 4b$. (*d*) $8ab - 9bc$. **8.** (*a*) $3x$. (*b*) $2a + 6$. (*c*) $a - 5$. (*d*) $-2y$. (*e*) $x - 8$. (*f*) $9x + 2$. **9.** (*a*) $-2x + 7y$. (*b*) $2a + 5b$. (*c*) $2a - 4b$. (*d*) $5x + 3y + 2z$. **10.** (*a*) $x^2 + 10x - 8$. (*b*) $3a^2 + 11$. (*c*) $-x^2 + 2xy + y^2$. (*d*) $5xy + yz - 6zx$. **11.** (*a*) $+15$. (*b*) $3a - 4$. **12.** (*a*) $+13$. (*b*) $-2x + y$. **13.** (*a*) -15. (*b*) $3x - 7y$. **14.** (*a*) $4a - 14b$. (*b*) $6x^2 - 11x + 9$. (*c*) 0. (*d*) $9y$. **15.** 0. **16.** $x + y - z$. **17.** 11. **18.** $2a - 2b$. **19.** $2y + 1$.

Page 216

4. (*a*) 6. (*b*) $-18{\cdot}288$. (*c*) $-31{\cdot}416$. (*d*) $+3{\cdot}6$. **5.** (*a*) $+48$. (*b*) -60. (*c*) -14. (*d*) -240. (*e*) $+12$. (*f*) -72. **6.** (*a*) -1. (*b*) $+1$. (*c*) $+4$. (*d*) -8. (*e*) $+16$. (*f*) -32. (*g*) $+9$. (*h*) $+27$. (*i*) $+81$. **7.** (*a*) -36. (*b*) $+\frac{1}{4}$. (*c*) $-0{\cdot}0021$. (*d*) $+1$. (*e*) $+72$. **8.** (*a*) 12. (*b*) 40. (*c*) -36. (*d*) -24. (*e*) $+3$. (*f*) $+68$.

Pages 218 and 219

7. (*a*) $+15ab^2c$. (*b*) $-10a^2$. (*c*) $+35xy$. (*d*) $-12ab$. (*e*) $+4x^2$. (*f*) $-27y^3$. (*g*) $-x^5$. (*h*) $+x^5$. (*i*) $-12x^3$. (*j*) $+\frac{1}{6}y^3$. **8.** (*a*) $-30a^2b^4$. (*b*) $+42x^4y^5$. (*c*) $+24x^6y$. (*d*) $-48a^5b^6$. (*e*) $-180x^2y^2$. (*f*) $+x^{13}y$. **9.** (*a*) $0{\cdot}1728a^2b^4$. (*b*) $2{\cdot}4021a^3bn$. **10.** (*a*) $+6$. (*b*) $+9$. (*c*) $+27$. (*d*) $+81$.

(*e*) +49. (*f*) +1. (*g*) −125. (*h*) −60. (*i*) +38. **11.** (*a*) $+2x^2$. (*b*) $+x^4$. (*c*) 0. (*d*) $+7x$. (*e*) $2x - 5$. (*f*) $+10x^2$. (*g*) $-11x$. (*h*) $+24x^2$. (*i*) $-x$. (*j*) $-7x$. (*k*) $+12x^2$. (*l*) $-576x^5$. **12.** (*a*) $-14a$. (*b*) $-12a$. (*c*) $-12x^2$. (*d*) $-3a$. (*e*) $-x - y$. (*f*) 0. (*g*) $-3a + 3b$. (*h*) $-8t^6$. (*i*) $+4a^6b^8$. (*j*) $-72a^7b^2$. **13.** (*a*) 0. (*b*) $-12x^6$. (*c*) $-9x^4$. (*d*) 0. (*e*) $x^2 + 9x^4$. (*f*) $16x^4$. (*g*) $10x^4$. (*h*) $7x^4$. (*i*) $49x^4$. **14.** (*a*) $15xy$. (*b*) $15a^2$. (*c*) $\frac{1}{2}\frac{5}{2}x^2y$. (*d*) $8{\cdot}75x^2$. (*e*) $21{,}875x^2y$. (*f*) $45xy$. **15.** $4{\cdot}5x^3$. **16.** (*a*) −15. (*b*) −45. (*c*) +60. (*d*) −45. (*e*) 900. (*f*) −270.

Page 222

5. (*a*) −6. (*b*) $-4\frac{1}{6}$. (*c*) $-\frac{7}{8}$. (*d*) +40. (*e*) $-\frac{1}{10}$. (*f*) $-1\frac{1}{3}$. **6.** (*a*) +400. (*b*) +0·0025. (*c*) −10. (*d*) −12·5. (*e*) −49. (*f*) +360. **7.** (*a*) $+12b$. (*b*) $-2\frac{1}{16}xy$. (*c*) $-1\frac{1}{16}$. (*d*) $-\frac{9}{10}z$. (*e*) $-\frac{1}{8}y$. (*f*) $+1\frac{1}{3}y$. **8.** (*a*) $+\frac{1}{2}$. (*b*) −5. (*c*) $-\frac{1}{6}$. (*d*) +4. (*e*) $-12\frac{1}{2}$. (*f*) $+1\frac{1}{18}$. **9.** (*a*) $15y$. (*b*) $10x$. (*c*) $5y$. (*d*) $3y$. (*e*) $1{\cdot}5x$. (*f*) x.

Pages 223 and 224

4. (*a*) $4ab$. (*b*) $-6a^2b^2$. (*c*) $+6ab^2$. (*d*) $-0{\cdot}4b^2$. (*e*) $70a^5$. (*f*) $2ab$. **5.** (*a*) $2a^2$. (*b*) $\frac{4}{3}d$. (*c*) 2. (*d*) 8. (*e*) $24r^2$. (*f*) $54y$. **6.** (*a*) a^6. (*b*) 1. (*c*) 1,000. (*d*) 10,000. (*e*) $2a^2$. (*f*) $+4a^4$. (*g*) πr. (*h*) r^2. (*i*) $1\frac{1}{3}rh$. **7.** (*a*) 15. (*b*) 240. **8.** 16. **9.** $4ab$. **10.** $5x$. **11.** $49x$.

Pages 225 and 226

4. (*a*) −6. (*b*) −27. (*c*) +27. (*d*) +7. (*e*) +36. (*f*) +16. (*g*) +81. **5.** (*a*) +10. (*b*) +5. (*c*) +4. (*d*) +19. (*e*) −8. **6.** (*a*) +36. (*b*) +9. (*c*) +256. (*d*) +25. (*e*) 0. (*f*) +13. **7.** (−6, −7); (−5, −5); (−4, −3); (−3, −1); (−2, +1); (−1, +3); (0, +5); (+1, +7); (+2, +9); (+3, +11); (+4, +13); (+5, +15); (+6, +17). **8.** (−3, +25); (−2, +22); (−1, +19); (0, +16); (+1, +13); (+2, +10); (+3, +7); (+4, +4); (+5, +1); (+6, −2). **9.** (−2, +9); (−1, +4); (0, +1); (+1, 0); (+2, +1); (+3, +4). **10.** (*a*) +20. (*b*) +9. (*c*) +2. (*d*) $+\frac{7}{8}$. (*e*) 0. (*f*) −1. (*g*) −1. (*h*) 0. (*i*) +5. **11.** (*a*) −5. (*b*) +1. (*c*) +13. (*d*) −5. (*e*) −35. (*f*) +19. **12.** (*a*) +44·6. (*b*) +23. (*c*) +5. (*d*) −40. **13.** (*a*) +20. (*b*) 0. (*c*) $-17\frac{7}{9}$. (*d*) −25. (*e*) −40.

Pages 227, 228 and 229

Review Exercise

6. 49. **7.** (*a*) $10x - 3y - 4z$. (*b*) $-5x^2 - 8x - 5$. (*c*) $-ab - 3ac$. (*d*) $-11x + 5y$. (*e*) $-5x^3 + 8x^2$. **8.** (*a*) $x - y + 2z$. (*b*) $5a - 2b$. (*c*) $5a - 2b$. (*d*) $3x - 2$. (*e*) $12 - x$. (*f*) $-9 - 2x$. **9.** $3x + 3y$. **10.** $11\frac{2}{3}x^3$. **11.** (*a*) $8x^3$. (*b*) $25x^2$. (*c*) $9x^2$. (*d*) $-27x^3$. (*e*) $+16x^4$. (*f*) +1. (*g*) 1. (*h*) 1. (*i*) $+9x^3y^2$. (*j*) $-1{\cdot}2$. (*k*) $10x$. (*l*) −2. (*m*) $-350x^3$. (*n*) 0·01. (*o*) 10,000. **12.** (*a*) 0·09. (*b*) 0·002. (*c*) 1·25. (*d*) 0·8.

(*e*) 0·0025. (*f*) 0·000064. (*g*) 0·0525. (*h*) 1·1025. (*i*) −0·1. (*j*) 0·0001. (*k*) 0·0001. (*l*) 4. **13.** (*a*) $\frac{11}{16}$. (*b*) $-\frac{9}{16}$. (*c*) $\frac{5}{128}$. (*d*) $\frac{1}{10}$. **14.** $40x^2$. **17.** (*a*) 71. (*b*) 40. (*c*) 19. (*d*) 8. (*e*) 6·65. (*f*) 7. (*g*) 35. (*h*) 64. (*i*) 103. **18.** 0·785; 1·57; 2·36; 3·14; 3·14; 7·07; 12·6; 19.6; 28·3; 38·5.

Pages 234 and 235

25. 6. **26.** 2. **27.** $-\frac{1}{3}$. **28.** 6. **29.** 12. **30.** 10. **31.** 8. **32.** 5. **33.** 6. **35.** 2. **36.** 17. **37.** $-\frac{5}{8}$. **41.** 5. **42.** $7\frac{3}{4}$. **43.** 14. **44.** 200.

Pages 236 and 237

10. 18. **11.** 210. **12.** $2\frac{2}{3}$. **13.** 7. **14.** $41\frac{1}{2}$. **15.** 16. **16.** 30. **17.** $\frac{A}{\pi h}$. **18.** $\frac{E}{R}$. **19.** $p - b - c$. **20.** $\frac{d}{v}$. **21.** $\frac{I}{pt}$. **22.** $\frac{b}{a}$. **23.** $\frac{b+d}{m}$. **24.** $c - am$. **25.** $\frac{k-b}{m}$. **26.** ac. **27.** $\frac{mn}{k}$. **28.** $\frac{am}{n}$. **29.** $\frac{b+d}{3}$. **30.** $90 - c$. **31.** $180 - b - c$. **32.** $\frac{2A}{h} - y$. **33.** $a - c - p$. **34.** $\frac{12b}{nw}$. **35.** $\frac{k-m}{n-1}$. **36.** $\frac{v - 16t^2}{t}$.

Pages 239, 240 and 241

6. 6. **7.** 7. **8.** 9. **9.** 8. **10.** 25, 26. **11.** 20, 21, 22. **12.** 17. **13.** 16. **14.** 440. **15.** $323\frac{1}{13}$. **16.** $9\frac{1}{2}$. **17.** 48, 48, 12. **18.** 24. **19.** 3′8″. **20.** 7, 32. **21.** 19. **22.** 6, 25. **23.** 24, 6. **24.** 5. **25.** 40, 46, 45. **26.** 2,375; 7,625. **27.** $1.81\frac{9}{11}$. **28.** 1,025; 3,475. **29.** 4.00. **30.** 55. **31.** $2{,}511\frac{3}{7}$, $12{,}857\frac{1}{7}$, $128{,}571\frac{3}{7}$. **32.** 0.5. **33.** 1,080. **34.** (*a*) $x = 15$. (*b*) $x = \frac{13}{3}$. (*c*) $x = 7$. (*d*) $x = 6$. (*e*) $x = 6$. (*f*) $x = -9$. **35.** Numbers are 17, 18, 19. **36.** Width is 18 ft. length is 20 ft. **37.** These sides are 20 ft. 26 ft. and 45 ft. **38.** Man and wife are each 32 years old. The son is 7 years old, daughter 5 years old.

Pages 241 and 242

Review Exercise

5. $5\frac{1}{2}$. **6.** 31, 32, 33, 34. **7.** 15, 17. **8.** 3,000; 9,000; 15,000. **9.** 96. **10.** 28, 53. **11.** 17. **12.** (*a*) $s = 18$. (*b*) $t = -8$. (*c*) $a = 20$. (*d*) $x = -16$. (*e*) $n = 3$. (*f*) $t = -4$. **13.** (*a*) $x = n - m$. (*b*) $y = \frac{p+6}{8}$. (*c*) $x = 2 - 3r$. (*d*) $y = \frac{b+c}{5}$. (*e*) $y = \frac{10}{3a}$. (*f*) $x = 5m$. **14.** The number is 8.

15. The boy received $48.00 as a birthday gift.

16. Each pencil cost 8 cents. **17.** The man worked 17 days.

Pages 244 and 245

6. (*a*) $4x + 10y$. (*b*) $3x^3 + 13x^2 - 10x + 20$. (*c*) $3a + 3b$. (*d*) $2x^2y^2$. (*e*) $a^2b - ab^2$. **7.** πy. **8.** (*a*) $2x - 7$. (*b*) $-7x - 14$. (*c*) $8a + 2$.

(*d*) $2z^2 - 4z$. **9.** (*a*) $-7x^2 - 24x + 21$. (*b*) $2x^3 - 3x^2 + 8x - 12$. (*c*) $4a - 3b + c$. (*d*) $y^3 + y^2 + 4$. **10.** $y^2 - x^2$. **11.** $6x - 9y + 32$. **12.** (*a*) $5a - 3b + 2c$. (*b*) $2ab$. **13.** $8x - 10$. **14.** $\pi d + 6$.

Pages 248 and 249

7. (*a*) $x^2 + 5x + 6$. (*b*) $y^2 - 7y + 10$. (*c*) $x^2 + 2xy + y^2$. (*d*) $6a^2 + 19a + 15$. (*e*) $4a^2 + 11a + 6$. (*f*) $2a^2 - 15a + 25$. **8.** (*a*) $7x^2 + 27x + 23$. (*b*) $-2x^2 - x + 7$. (*c*) $4xy$. (*d*) $13x^2 - 17y^2$. **9.** $2bx + 10b^2$. **10.** $4x^2 + 9xy - 12y^2$. **11.** (*a*) $2x^2 - 4x + 5$. (*b*) $-5n + 5$. (*c*) $y^2 + 4y - 1$. (*d*) $-2x^2 + 10x - 15$. **12.** (*a*) $-3x$. (*b*) $6x^2 - 22xy + 6y^2$. (*c*) $32x^2 - 48xy - 18y^2$. **13.** $8x^2 - 16x - 120$. **14.** $3x^2 + 6xy + 3y^2$. **15.** $\pi a^2 + 2\pi ab + \pi b^2$. **16.** $\frac{1}{2}x^2 - 2$.

Page 251

75. $5x^2 + 30$. **76.** $2x^2 + 6x - 8$. **77.** $2xy$. **78.** $2a^2 + 2b^2$. **79.** $10x^2 + 11xy - 13y^2$. **80.** $a^2 + 14ab - 14b^2$. **81.** $3x^2 - 10x + 17$.

Pages 252 and 253

10. $2a + b$. **11.** $-2x^2 + 5x + 6$. **12.** $-a + 2b$. **13.** $2x^2 - 4x - 3$. **14.** $30a + 20b$. **15.** $5a - 6$. **16.** $-a^2 + a + 1$. **17.** $x + 1$. **18.** $2x - 1$. **19.** $2y$. **20.** 2. **21.** $5c - 5a$. **22.** $2a + 8b$. **23.** $4 - 3b$. **24.** $x - 2$.

Page 254

1. (*a*) $2x + 1$. (*b*) $x + 7$. (*c*) $x + 3$. (*d*) $3a + 2$. (*e*) $3n + 1$. (*f*) $x - 4$. **2.** $x^2 + x + 4$. **3.** $x + y$. **4.** $5x + 2$. **5.** $a + 3$. **6.** $x + 2y$. **7.** $1 - 2x$. **8.** $m - 3$. **9.** $x^2 + x - 4$. **10.** $x^2 - 2x - 2$. **11.** $2k + 1$.

Pages 255 and 256

Review Exercise

6. (*a*) $5a - 15$. (*b*) $6a - 10$. (*c*) $x - 7y$. (*d*) $-3b$. **7.** $4x^2 + 28x + 48$. **8.** (*a*) $3x^2 + 6x - 9$. (*b*) $5x^3 - 10x^2 + 5x$. (*c*) $4a^2 - 10ab - 6b^2$. (*d*) $-12x^2 + 75y^2$. **9.** (*a*) $2x^2 + 10x + 14$. (*b*) 9. (*c*) $12x$. (*d*) $3a^2 + 11ab - 46b^2$. (*e*) $5x^2 + 18x + 44$. (*f*) $-x^2 + 8x - 10$. **10.** (*a*) 3. (*b*) 2. (*c*) 2x + 12y. (*d*) 2x − 4y. **11.** (*a*) $3x^2 - 8x + 5$. (*b*) $9x + 11$. (*c*) $6x^2 + 24x - 16$. **12.** (*a*) $a^2 - ab + b^2$. (*b*) $x^2 + 2x + 1$. (*c*) $x^2 - 2x + 1$. **13.** (*a*) $y + 2$. (*b*) $3x + 2$. (*c*) $y - 1$. (*d*) $y + 2$. (*e*) $a + 3$. (*f*) $x + 2y$. **14.** length of side is $(2m + 3)$ units. **15.** length of altitude $(x - 3)$ ins.

Page 260

31. $(a + d)(b + c)$. **32.** $(a - b)(x - y)$. **33.** $(1 + q)(2 - p)$. **34.** $(a + b^2)(a^2 + b)$. **35.** $(m + 4)(x + 2)$. **36.** $(a + b)(x - y)$. **37.** $(x - 1)(x - 1)(x + 1)$. **38.** $(x + 2)(3x^2 + y)$. **39.** $(m + n)(m^2 + n)$.

40. $(y + 2)(2x - 3)$. **41.** $(m - p)(n - q)$. **42.** $(x + 3)(ab - 2)$.

Page 262

49. $2(x + 3)(x + 1)$. **50.** $2(x + 3)(x + 2)$. **51.** $a(x + 6)(x + 1)$.
52. $2x(x + 6)(x + 3)$. **53.** $5(y + 2z)(y - 3z)$. **54.** $3(z + 4)(z - 2)$.
55. $m(r - 3)(r - 2)$. **56.** $5(n - 5)(n - 6)$. **57.** $(2x + 3)(2x + 3)$.
58. $(2a + 1)(a + 2)$. **59.** $(3b - 1)(b + 3)$. **60.** $(7 + 3y)(7 + 3y)$.
61. $(3x - 2y)(2x - 3y)$. **62.** $(3y - 5)(3y - 5)$. **63.** $(4x - 5)(5x + 6)$.
64. $x^2(3x + 5)(2x - 1)$.

Page 264

25. $2(a + b)^2$. **26.** $2(a + b)(a - b)$. **27.** $3(x + y)(x - y)$. **28.** $5(a - b)^2$.
29. $a(x + y)(x - y)$. **30.** $a(m + 2)^2$. **31.** $5(1 - 3n)(1 + 3n)$.
32. $a^2(a - 2b)(a + 2b)$. **33.** $(x - y + 3)(x + y + 3)$. **34.** $(a + b + c)^2$.
35. $(x + y - z)(x + y + z)$. **36.** $(-a - 3b)(5a + 3b)$. **37.** 49. **38.** 99.
39. 197. **40.** 25. **41.** 2,601. **42.** 5,041. **43.** 961. **44.** 3,721.
45. 1,521. **46.** 2,401. **47.** 7,744. **48.** 5,929. **49.** $a + 3b$.
50. $5a + 8b$. **51.** (*a*) a^2. (*b*) $2ab$. (*c*) y^2. (*d*) 1. (*e*) 2.

Pages 266 and 267

13. $\frac{2ab}{3c^2}$. **14.** $\frac{1}{km}$. **15.** $\frac{y}{3x}$. **16.** $\frac{-2}{ab}$. **17.** $\frac{3}{14}$. **18.** $\frac{1}{2}$. **19.** $4\frac{5}{9}$. **20.** $\frac{x+y}{y}$.

21. $\frac{1}{2}$. **22.** $\frac{a-2b}{a+2b}$. **23.** $\frac{x+1}{x-1}$. **24.** $\frac{x}{w}$. **25.** $\frac{a}{a-1}$. **26.** $\frac{1}{m-n}$. **27.** $\frac{1}{m+n}$.

28. $\frac{m+3}{m}$. **29.** $\frac{a+b}{a-b}$. **30.** $\frac{a+5}{a+3}$. **31.** $\frac{x-3}{x+3}$. **32.** $\frac{x}{y}$. **33.** $\frac{a+2}{a+3}$. **34.** -1.

35. 1. **36.** $x + 4$. **37.** $-(x + 4)$. **38.** $\frac{y}{x+8}$. **39.** -1. **40.** 1. **41.** $\frac{ax}{5by}$.

42. $\frac{a^2b}{6xy}$. **43.** $\frac{2}{3}$. **44.** 1. **45.** $\frac{x+1}{2}$. **46.** $\frac{5}{x}$. **47.** $\frac{a-1}{a-2}$. **48.** $-\frac{5}{6}$. **49.** $\frac{x^2}{y^2}$.

50. $\frac{a+b}{(a-b)(a+2b)}$. **51.** $\frac{x+1}{x^2(x-1)}$.

Pages 268 and 269

1. $\frac{2}{x}$. **2.** $\frac{9}{2x}$. **3.** $\frac{19x}{12}$. **4.** $\frac{x+10}{5x}$. **5.** $\frac{8m+7}{2m}$. **6.** $\frac{5x+4}{6}$. **7.** $\frac{5d-3}{3d}$.

8. $\frac{14x-3}{6}$. **9.** $\frac{11n-8}{3n}$. **10.** $\frac{4d^2+7d-12}{3d}$. **11.** $\frac{3-2x}{6x}$. **12.** $\frac{x+2}{2}$.

13. $\frac{14-7x}{3x}$. **14.** $\frac{3x-3}{x}$. **15.** $\frac{12x-1}{4}$. **16.** $\frac{2xy-4}{x^2y^2}$. **17.** $\frac{7x^2+28x-12}{28x^2}$.

18. $\frac{x-4}{18}$. **19.** $\frac{q-6.}{30}$. **20.** $\frac{-5n-27}{12}$. **21.** $\frac{15x+4}{20}$.

22. $\frac{13-2x}{2x}$. **23.** $\frac{-31x}{90}$. **24.** $\frac{x}{6}$.

Page 270

1. $\frac{-x^2+10x-8}{2x(x-4)}$. **2.** $\frac{x-14}{3x(x-2)}$. **3.** $\frac{3x^2-3x+5}{x^2}$. **4.** $\frac{-2x}{x^2-16}$.

5. $\frac{d^2-7d-12}{d^2-9}$. **6.** $\frac{-x^2-9x+2}{(x^2-1)(x-2)}$. **7.** $\frac{-4x+5x-36}{2x-6}$.

8. $\frac{-6x^3-6x^2-51x+21}{2x(x^2-9)}$. **9.** $\frac{-2x+22}{(x-2)(x+7)^2}$. **10.** $\frac{6x-5}{(x-1)(x-1)(x+1)}$.

11. $\frac{8x}{x^2-4}$. **12.** $\frac{3x^2-6x+9}{(x-3)(x+3)^2}$. **13.** $\frac{6a-9}{2a+3}$.

14. $\frac{2x^2+2xy-3y^2}{y(x+y)^2}$. **15.** $\frac{c^2+1}{c^2+2}$. **16.** $\frac{4x+5}{(3x-2)(3x+4)}$.

Pages 271 and 272

4. $(b-a)x-c$. **5.** $m-x(p+q)$. **6.** $(a-c)x-b-d$. **7.** $(ab-bc)x+(3a+5ab)$. **8.** $(a+b)x^2+cx+d$. **9.** $mx^2+(n-5)x+(c-3)$. **10.** $(a+d)x^3+(b-g)x^2+(c-f)x$. **11.** $(f-a)x^2+(c-b)x+g$.

12. $\frac{b}{a}$. **13.** $\frac{p}{m+n}$. **14.** $\frac{c}{a-5}$. **15.** $\frac{d}{a+b+3}$. **16.** $\frac{p}{2n+2}$. **17.** $\frac{k}{1+n}$.

18. $\frac{b-2a}{a-b}$. **19.** $\frac{bc-ad}{c-d}$. **20.** $\frac{w}{w-k}$. **21.** $\frac{2A}{y+z}$. **22.** 625. **23.** 1,225. **24.** 2,025. **25.** 3,025. **26.** 621. **27.** 1,224. **28.** 9,016. **29.** 1,591. **30.** 899. **31.** 4,896. **32.** 8,096. **33.** 8,099. **34.** 2,475. **35.** 32. **36.** 103. **37.** 1,040. **38.** 9. **39.** 7. **40.** 10. **41.** 15. **42.** 11. **43.** 3.

Pages 273 and 274

19. $3(1-3n)(1+3n)$. **20.** $4(a-5)(a+5)$. **21.** $4(a^2+4)$. **22.** $5(a^2-10)$. **23.** $3(x-1)^2$. **24.** $a(x-y)(x+y)$. **25.** $\frac{1}{2}h(x+y)^2$. **26.** $\pi(R-r)(R+r)$. **27.** $\pi r(h+2r)$. **28.** $2\cdot 4a(a-2)$. **29.** $(a-b)(a+b)(a^2+b^2)$. **30.** $(a-1)(a+1)(a^2+1)$. **31.** $(a-2)(a+2)(a^2+4)$. **32.** $(a-3)(a+3)(a^2+9)$. **33.** 12. **34.** 8. **35.** 16. **36.** $x+3$. **37.** (*a*) ± 5. (*b*) $\pm 4ab$. (*c*) $\pm(4a+1)$. **38.** $(x+y)(a+b)$. **39.** $(x+y)(a+1)$. **40.** $(p-q)(a+b+c)$. **41.** $2(1-a+b)(1+a-b)$. **42.** $(a-b-c)(a+b+c)$. **43.** $(3-a+b)-(3+a-b)$. **44.** $(x-a+b)(x+a-b)$. **45.** $(4-x+y)(4+x-y)$.

46. (*a*) $\frac{3xy-3x^2-2y^2}{xy(x-y)}$. (*b*) $\frac{3x^2+2x-108}{(x+6)^2}$.

(*c*) $\frac{m^2 + mn - n^2}{(m - n^2)}$. (*d*) $\frac{-3x^2 + 14x + 21}{3(x + 3)(2x + 3)}$.

47. The perimeter is $\frac{8x^2 + x - 5}{x^2 - 1}$ units. **48.** $-(p - 5)(q + 6)$.

49. $(n + 3)(a - b)$. **50.** $(5d - 6c)(4d + 9c)$. **51.** $an(3 - a)(3 + a)$.

52. $(a + b + 1)(a + b + 1)$. **53.** $(2x + 1)(x - 2)(x + 2)$.

54. $(x - y - 3)(x + y - 3)$. **55.** $\frac{x + 2}{m - n}$. **56.** $\frac{5x}{2r + 6s}$.

57. $\frac{-(6x + 1)(2x + 1)}{2x + y}$.

Page 277

6. (*a*) 7·5. (*b*) −1·5. (*c*) 3·55. (*d*) 3. **7.** (*a*) $18 - 3x$. (*b*) $\frac{15 + 3x}{4}$.

(*c*) $x - 7$. (*d*) $\frac{5x - 13}{4}$. **8.** $\frac{E}{R}$. **9.** $2s - b - c$. **10.** $\frac{15 - b}{a}$. **11.** $\frac{b - c}{a}$.

12. $\frac{n + p + q}{m}$. **13.** $\frac{b}{c - a}$. **14.** $\frac{2A - hy}{h}$. **15.** 80 cents **16.** (*a*) 4. (*b*) 6.

Pages 278, 279 and 280

1. (*a*) 9. (*b*) $3\frac{6}{7}$. (*c*) −7. (*d*) −5. **2.** (*a*) −3. (*b*) −2. (*c*) $\frac{1}{2}$.
3. (*a*) 0. (*b*) 1. (*c*) $\frac{2}{3}$. (*d*) 2. **4.** (*a*) −2. (*b*) 15. (*c*) 7.
5. (*a*) 14. (*b*) 6. (*c*) −8. (*d*) 2. **6.** 27, 19, 54. **7.** 8, 11, 11.
8. 17, 18, 19. **9.** 17, 9. **10.** 21, 4. **11.** 11, 22, 27. **12.** 19. **13.** 40, 48.
14. 8000, 2000. **15.** 4000, 1000. **16.** 34. **17.** 21, 63. **18.** $\frac{c}{\pi}$.
19. 7, 9, 11. **20.** $-\frac{1}{3}$ **21.** 16, 48. **22.** $6{,}666\frac{2}{3}$. **23.** 3. **24.** 6. **25.** 8·1.
26. 24. **28.** 19, 21. **29.** 12. **30.** 0·26. **31.** (*d*) $A = \pi R^2 - \pi r^2$.

Pages 283, 284 and 285

19. 7. **20.** 17. **21.** −3. **22.** 14. **23.** 35. **24.** 0. **25.** $1\frac{2}{5}$.
26. $13\frac{2}{3}$. **27.** $\frac{4}{11}$. **28.** 0. **29.** 14. **30.** $8\frac{4}{7}$. **37.** 8. **38.** 9.
39. 0·8. **40.** 5. **41.** 60. **42.** $\frac{2A}{b}$. **43.** $\frac{2A}{h} - 5$. **44.** $\frac{5F - 160}{9}$.
45. 4·90. **46.** $2\frac{6}{7}$. **49.** $14\frac{2}{5}$. **50.** 24. **51.** $-5h$. **52.** 53.
53. −1. **54.** 3. **55.** $m = 9$. **56.** $x = -\frac{1}{3}$. **57.** $a = -1$.

58. $d = \frac{L - a}{n - 1}$. **59.** $i = \frac{E}{R + r}$; $R = \frac{E - ir}{i}$.

Pages 286 and 287

Review Exercise

5. −1. **6.** 150. **7.** −2. **8.** −91. **9.** −3. **10.** 48. **11.** $4\frac{1}{2}$.
12. $\frac{2A}{h} - y$. **13.** $\frac{3V}{\pi r^2}$ **10.** $2k$. **15.** −36. **16.** 5. **17.** 16,000,
$1{,}333\frac{1}{3}$, $6{,}666\frac{2}{3}$. **18.** 20. **19.** (*a*) 7. (*b*) 1. (*c*) 42. (*d*) 1.

20. (*i*) $\frac{RD}{r}$ (*ii*) $\frac{rd}{R}$. **21.** $\frac{9}{5}C + 32$. **22.** 7. **23.** $\frac{a^2+b^2+c^2}{2a}$
24. 11.625. **26.** $\frac{1}{4}$. **27.** $P = \frac{n}{S-L}$. **28.** $x = \frac{c}{S+P}$.
29. (*a*) $x = \frac{3}{8}$. (*b*) $m = -3$. (*c*) $c = -\frac{5}{6}$.
(*d*) $x = \frac{-46}{3}$. (*e*) $x = -4$.
30. $4\frac{2}{3}$ ohms and $7\frac{1}{3}$ ohms. **31.** 10 gallons and 30 gallons.
32. (*a*) $(2x^2 - 27x + 90)$ sq. ins. (*b*) 54 sq. in.
33. Increase of $4\frac{4}{9}°$.

Pages 295 and 296

1. (4, 1). **2.** (2, 3). **3.** (5, — 1). **4.** (— 2, — 3). **5.** (— 2, 3).
6. (2, 1). **7.** (1, 2). **8.** $(2\frac{1}{2}, 3)$. **9.** $(1\frac{1}{3}, 2\frac{1}{4})$. **10.** (0, 0).

Pages 298, 290 and 300

1. $(4\frac{1}{2}, \frac{1}{2})$. **2.** $(1, \frac{1}{4})$. **3.** $(1, \frac{1}{3})$. **4.** $(\frac{1}{3}, \frac{1}{4})$. **5.** (2,1).
6. (1, 2). **7.** $x = 2, y = 3$. **8.** $x = -3, y = -1$. **9.** $x = 3$, $y = 5$. **10.** $a = -2, b = 3$. **11.** $x = \frac{1}{2}, y = 6$. **12.** $x = 2$, $y = 3$. **13.** $a = 0, b = -2$. **14.** $x = -1, y = 6$. **15.** $m = 6$, $n = 5$. **16.** $x = 2, y = 3$. **17.** $x = 2, y = 14$. **18.** $x = -4$, $y = -\frac{14}{3}$. **19.** 17, 18. **20.** 5, — 8. **21.** 17. **22.** 250, 0.98.
23. 4, 3. **24.** 3.10, 1.40. **25.** $\frac{13}{15}$. **26.** 13, 7. **27.** 12, 2. **28.** 16, 9.
29. 10, 12. **30.** 2, 0.1. **31.** 3, 4. **32.** 15, — 5. **33.** 32. **34.** 80, 12.
35. 25, 5. **36.** $1\frac{1}{2}$, 360. **37.** (*c*) 1 p.m. (*d*) (*i*) 3. (*ii*) 0.

Page 304

4. (*a*) $\frac{21}{64}$. (*c*) $\frac{53}{64}$. (*e*) $\frac{47}{64}$. **5.** (*a*) $2\frac{6}{32}$. (*c*) $5\frac{2}{32}$. **6.** (*a*) $\frac{22}{64}$. (*c*) $\frac{4}{64}$. **7.** (*a*) $\frac{55}{64}$. (*c*) $\frac{43}{64}$. **8.** (*a*) 0·005. (*c*) 0·003. **9.** (*a*) $8\frac{42}{64}$. (*b*) $11\frac{12}{64}$. (*c*) $3\frac{10}{64}$. **10.** (*a*) 13′ $7\frac{3}{8}$″. (*b*) 8′ $6\frac{3}{4}$″. (*c*) 9′ $2\frac{1}{8}$″.

Page 313

1. 5,024. **2.** 4,710. **3.** 2,198. **4.** (*a*) 5,200. (*b*) 4,600. (*c*) 5,400. (*d*) 8,500. **5.** (*a*) 1,911. (*b*) 2,866. **6.** 255. **7.** 672. **8.** (*a*) 5,700. (*b*) 780. (*c*) 700. (*d*) 9,600. **9.** 7. **10.** 18. **11.** 10·6. **12.** (*a*) 6. (*b*) 22. (*c*) 3. (*d*) 32. **13.** 19. **14.** 6.

Page 315

1. 57·3. **2.** 713. **3.** 170. **4.** 159. **5.** 229. **6.** (*a*) 19·0 sec. (*b*) 27·5 sec. (*c*) 68·7 sec. **7.** 0·234.

Page 316

1. 152·9. **2.** 52·3. **3.** 1,274. **4.** (*a*) 131. (*b*) 127. (*c*) 336. **5.** 76·4. **6.** 764. **7.** 85. **8.** 61·2. **9.** 733. **10.** (*a*) 96. (*b*) 115. (*c*) 287. (*d*) 1,911. (*e*) 229.

Pages 317 and 318

1. (*a*) 3,825. (*b*) 52·5. (*c*) 52·5. (*d*) 85·9. **2.** (*a*) 3,333. (*b*) 82. (*c*) 480. (*d*) 1,173. **3.** 15. **4.** 5·8. **5.** 4,500. **6.** 190. **7.** 1,333; 800; 1,600; 427. **8.** 100. **9.** 17.

Page 319

1. (*a*) 0·0225. (*b*) 0·0114. (*c*) 0·0156. **2.** (*a*) 0·05. (*b*) 0·03125. (*c*) 0·0096. **3.** 30. **4.** 405. **5.** 0·0179. **6.** (*a*) 9. (*b*) 3. (*c*) 30. (*d*) 12. **7.** 10·0. **8.** 7·5.

Pages 320, 321 and 322

6. 12:7. **9.** (*a*) 8:7. (*b*) 11:10. (*c*) 2:1. **10.** (*a*) 2:3. (*b*) 1:5. (*c*) 2:1. **11.** 40:7. **12.** 1:5. **13.** 8:15. **14.** 8:13. **15.** (*a*) 1:2. (*b*) 1:2. (*c*) 1:4. (*d*) 1:2. **16.** (*a*) 2:3. (*b*) 2:3. (*c*) 4:9. (*d*) 8:27. **17.** (*a*) 3:2. (*b*) 2:3. **18.** 6:1. **19.** 200:1. **20.** (*a*) $\frac{1}{2}$. (*b*) $\frac{2}{5}$. (*c*) $\frac{1}{8}$. (*d*) $\frac{1}{4}$. (*e*) $\frac{1}{4}$. (*f*) $\frac{3}{8}$. (*g*) $\frac{5}{24}$. **21.** 7·2; 0·35; 1·4; 0·94; 0·88; 8·9; 0·40; 2·3; 16·5; 19·2. **22.** 9:5.

Pages 323 and 324

1. (*a*) 700. (*b*) 840. (*c*) 455. (*d*) 175. **2.** 4·00. **3.** 27. **4.** (*a*) $3\frac{1}{2}$. (*b*) $2\frac{1}{4}$. (*c*) 19·8. (*d*) 56. (*e*) $3\frac{1}{3}$. (*f*) 12. (*g*) 72. (*h*) $\frac{3}{4}$. **5.** 5. **6.** (*a*) 6·928. (*b*) 1·732. (*c*) 4·33. (*d*) 8·66. (*e*) 0·108. (*f*) 0·0724. **7.** (*a*) 9·24. (*b*) 2·31. (*c*) 12·0. **8.** (*i*) 0·8. (*ii*) 2·56. (*iii*) 0·56. **9.** (*i*) 72. (*ii*) 9·6. (*iii*) 63·6. **10.** (*i*) 0·3. (*ii*) 0·4. (*iii*) 0·75. **11.** (*i*) 8. (*ii*) $13\frac{1}{3}$. (*iii*) 24.

Pages 325 and 326

1. 50. **2.** (*a*) $13\frac{1}{2}$. (*b*) 8. (*c*) 4. **3.** (*a*) $44\frac{4}{9}$. (*b*) 750. **4.** 50. **5.** $18\frac{6}{13}$. **6.** (*a*) 8. (*b*) 60. **7.** (*a*) $8\frac{1}{3}$. (*b*) $5\frac{1}{4}$. (*c*) $6\frac{1}{4}$. **8.** (*a*) 500. (*b*) $208\frac{1}{3}$. (*c*) 312·5. **9.** (*a*) 250. (*b*) 150. (*c*) 400.

Pages 328 and 329

1. (*a*) 144. (*b*) 90. (*c*) 24. (*d*) 40. **2.** (*a*) $333\frac{1}{3}$ (*b*) 750. (*c*) $119\frac{1}{9}$ (*d*) $171\frac{3}{7}$ **3.** $178\frac{4}{7}$ **4.** 222. **6.** S — 60, L — 80. **7.** d — 68, f — 40. **8.** $228\frac{4}{7}$. **9.** (*a*) 50. (*b*) 160.

Pages 331 and 332

1. 960. **2.** 80. **3.** 7·5. **4.** 9·6. **5.** (*a*) 144. (*b*) 4. **7.** 320. **8.** 3,360; 1,400; $583\frac{1}{3}$. **9.** (*a*) 375; 1,500; 6,000. (*b*) 2,880; 1,200; 436. **10.** (*a*) 1,600. (*b*) 10.

Pages 333 and 334

1. 0·600. **2.** 0·504. **3.** 0·052. **4.** 1·152. **5.** 0·875; 1·125. **6.** 0·5; 2·016. **7.** 2·116. **8.** 1·797. **9.** 0·624; 0·598; 0·599; 0·602; 0·623; 0·631. **10.** 0·125; 0·250; 0·625; $\frac{n}{8}$. **11.** 0·239; 0·299; 0·375; 0·420; 0·539; 1·260; 2·615.

Pages 335 and 336

1. $\frac{1}{8}$. **2.** 0·125. **3.** $\frac{7}{16}$. **4.** 0·081. **5.** 0·350. **6.** 0·352. **7.** (*a*) 0·365. (*b*) $\frac{15}{16}$. **8.** 0·057. **9.** 0·168. **10.** $\frac{9}{16}$. **11.** $\frac{3}{8}$, $\frac{9}{16}$. **12.** $\frac{1}{4}$.

Pages 336 and 337

1. 1,650. **2.** $1\frac{1}{14}$. **3.** 238. **4.** 1,465. **5.** 107. **6.** 37. **7.** 11·3; $\frac{1}{\sqrt{2}}$. **8.** $\frac{1}{\sqrt{2}}$. **9.** 1·3. **10.** 67·5. **11.** 560. **12.** 792. **13.** 0·6. **14.** 0·325. **15.** 0·2. **16.** $\frac{7}{24}$.

Pages 339 and 340

11. (*a*) 61° 8′. (*b*) 9° 36′. **12.** 48° 41′. **13.** 74° 42′. **14.** (*a*) 53°. (*b*) 76° 15′. (*c*) 4° 30′. (*d*) 9° 49′ 30″. (*e*) 79° 17′. (*f*) 80° 14′ 15″. **15.** (*a*) 22° 30′. (*b*) 5° 15′. (*c*) 7° 50′. (*d*) 43° 37′ 30″. (*e*) 8° 15′ 20″. (*f*) 65° 12′ 39″. **16.** 32° 42′; 17° 24′. **17.** (*a*) 45°. (*b*) 97° 50′. (*c*) 9° 40′. (*d*) 144° 14′. (*e*) 14° 31′ 50″. (*f*) 124° 12′ 1″. **19.** 121° 27′; 58° 33′. **20.** 151° 15′; 28° 45′. **21.** 15° or 165°. **22.** (*a*) 141° 14′ 30″. (*b*) 90°. **23.** 136° 55′ 49″. **24.** 105. **25.** 667. **26.** 38° 5′. **27.** (*a*) 205. (*b*) 236.

Pages 343 and 344

5. $\frac{11}{16}$. **6.** 1. **7.** $\frac{9}{16}$. **8.** 1. **9.** $1\frac{1}{2}$. **12.** $3\frac{1}{4}$. **14.** (*a*) 90.

Pages 347 and 348

5. 1271. **6.** 7812. **7.** 15·6. **8.** 13·2. **9.** (*b*) (*i*) 10. (*ii*) 3·6. (*iii*) 6·4. (*iv*) 4·8. **10.** $2\frac{2}{3}$; $5\frac{1}{3}$.

Pages 349 and 350

7. 70° 23′. **8.** 137° 27′ 41″. **9.** 64° 18′. **13.** 96° 3′. **15.** 120°. **17.** 1,080. **18.** (*a*) 900. (*b*) 1,260. (*c*) 1,440. (*d*) 1,800. (*e*) 2,340. (*f*) 2,520. **19.** $180n - 360$. **20.** 360. **22.** (*a*) 90. (*b*) 72. (*c*) 60. (*d*) 45. (*e*) 36. (*f*) $22\frac{1}{2}$. **23.** (*a*) 108. (*b*) 120. (*c*) 135. (*d*) 162. **24.** $180 - \frac{360}{n}$. **26.** 12. **27.** (*a*) 10. (*b*) 15. (*c*) 16.

Pages 351 and 352

2. (*a*) 1·3. (*b*) 5·8. **3.** (*a*) 30°; 90°. **4.** $2\frac{3}{16}$. **5.** $1\frac{1}{8}$. **6.** (*a*) 1,039. (*b*) 93·5. (*c*) 8,147.

Pages 358 and 359

1. (*a*) 180. (*b*) 210. (*c*) 504. (*d*) 540. **2.** (*a*) 60. (*b*) 192. (*c*) 120. (*d*) 660. **3.** (*a*) 28·6. (*b*) 13·2. (*c*) 3,923. **4.** 558. **5.** (*a*) 8·31. (*b*) 83·1. **6.** 3. **7.** 60. **8.** 24. **9.** 16; 120. **10.** (*a*) 1·73. (*b*) 43·3.

Pages 360 and 361

1. (*a*) 17·3. (*b*) 3·90. (*c*) 2·71. **2.** (*a*) 0·0866. (*b*) 0·0361. (*c*) 0·0541. (*d*) 0·1443. **3.** (*a*) 15·6. (*b*) 27·7. (*c*) 2·71. (*d*) 7·37. **4.** (*a*) 0·0325. (*b*) 0·0500. (*c*) 0·0812. (*d*) 0·1443. **5.** (*a*) 4·0. (*b*) 10·0. (*c*) 1·15. **6.** (*a*) 3·46. (*b*) 0·866. (*c*) 1·30. (*d*) 2·17. **7.** (*a*) 6·0. (*b*) 3·46. (*c*) 0·866. **8.** (*a*) 2·60. (*b*) 23·4. (*c*) 260. (*d*) 78·6. **9.** 260.

Pages 362 and 363

1. 84. **2.** 768. **3.** 30. **4.** 480. **5.** (*a*) 750. (*b*) 23·1. **6.** (*a*) 360. (*b*) 24·8, 20·0, 28·8. **7.** (*a*) 7·48. (*b*) 32·7. (*c*) 24·0. (*d*) 1,008. **8.** 9. **9.** (*a*) 4. (*b*) 9. (*c*) $\frac{1}{4}$. **10.** (*a*) 138. (*b*) 180. **11.** (*a*) 360. (*b*) 32.

Pages 364 and 365

1. 5. **2.** (*a*) 0·7. (*b*) 1·4. (*c*) 1·1. (*d*) 7·2. **3.** (*a*) 1·7. (*b*) 3·7. (*c*) 3·5. (*d*) 6·6. **4.** 7·1. **5.** 4. **6.** 144. **7.** (*a*) 17·0. (*b*) 25·5. (*c*) 19·1. (*d*) 23·0.

Page 366

1. (*a*) 7·07. (*b*) 14·1. (*c*) 1·41. (*d*) 2·83. (*e*) 4·24. (*f*) 35·4. **2.** (*a*) 0·71. (*b*) 0·35. (*c*) 0·88. (*d*) 1·77. **3.** (*a*) 0·10. (*b*) 0·05. (*c*) 0·13. (*d*) 0·26. **4.** (*a*) 2·83. (*b*) 10·6. (*c*) 14·1. (*d*) 0·707. (*e*) 1·41. (*f*) 0·707x. **5.** (*a*) (i) 0·35. (ii) 0·53. (iii) 1·06. (iv) 2·47. (*b*) (i) 0·07. (ii) 0·11. (iii) 0·22. (iv) 0·51. **6.** $1\frac{2}{3}\frac{1}{2}$.

Pages 367 and 368

1. (*a*) 12. (*b*) 18·4. (*c*) 25·5. (*d*) 60. **2.** 2 : 3. **3.** 5·2. **4.** 4·2. **5.** 6·3. **6.** (*a*) 0·14. (*b*) 0·31. **7.** 11·8. **8.** (*a*) 5·50. (*b*) 12·8. (*c*) 75·4. (*d*) 19·4. **10.** (*b*) $2\frac{1}{2}$. **11.** (*a*) 2·5. (*b*) $2\frac{1}{8}$. (*c*) $2\frac{1}{8}$. (*d*) $2\frac{1}{2}$. (*e*) $2\frac{1}{2}$.

Pages 368 and 369

Review Exercise

1. 330. **2.** 756. **3.** 47·3; 379. **4.** 10·4. **5.** 27·7. **6.** 0·0541. **7.** $1\frac{1}{2}$. **8.** 127. **9.** 0·53. **10.** $5\frac{1}{2}$; 28·8. **11.** 28. **12.** 7·85.

Pages 371 and 372

1. (*a*) 30. (*b*) 4·78. **2.** (*a*) 47·1. (*b*) 100. (*c*) 188. **3.** (*a*) 25. (*b*) 550. **4.** 1,159. **5.** (*a*) 204. (*b*) 283. **6.** 1,758.

Pages 372 and 373

1. (*a*) 236. (*b*) 75·4. (*c*) 170. **2.** 1,507. **3.** 1·2. **4.** 898. **5.** (*a*) 15. (*b*) 1,005. **6.** 55. **7.** 1 : 24. **8.** 1,017. **9.** 0·0440. **10.** 129,500.

Page 374

1. 384. **2.** (*a*) $8\frac{1}{2}$. (*b*) 136. **3.** 12. **4.** 297. **5.** 216. **6.** 41. **7.** 40. **8.** (*a*) 61. (*b*) 62. (*c*) 2,684. **9.** 384. **10.** 864.

Pages 375 and 376

1. (*a*) 128. (*b*) 96·9. (*c*) 6,720. **2.** 160. **3.** 120. **4.** 560. **5.** 756. **6.** (*a*) 12. (*b*) 400. **7.** 13,856. **8.** 139. **9.** 6. **10.** 18.

Page 377

1. (*a*) 113. (*b*) 452. (*c*) 314. (*d*) 1,256. (*e*) 78·5. (*f*) 707. **2.** 314. **3.** 201. **4.** 1,848. **5.** 75·4. **6.** 530. **7.** 170. **8.** (*a*) 4. (*b*) 50·2. **9.** 1,809. **10.** 5..

Page 378

1. (*a*) 0·523. (*b*) 4·19. (*c*) 14·1. (*d*) 382. **2.** (*a*) 785. (*b*) 523. **3.** 0·418. **4.** 11·3. **5.** 11·2. **6.** 244. **7.** 622. **8.** (*a*) 8 : 1. (*b*) 27 : 1. (*c*) 512 : 1. **9.** 81.

Page 379

Review Exercise

1. (*a*) 22·6. (*b*) 603. (*c*) 1,514. **2.** (*a*) 13·4. (*b*) 466. (*c*) 576. **3.** (*a*) 28·3. (*b*) 14·1. **4.** 65·4. **5.** 21·7. **6.** 11·2.

Page 382

4. (*a*) 6. (*b*) (i) $\frac{3}{5}$. (ii) $\frac{4}{5}$. (iii) $\frac{3}{4}$. (iv) $\frac{4}{5}$. (v) $\frac{3}{5}$. (vi) $\frac{4}{3}$. **5.** (i) $\frac{24}{25}$; $\frac{7}{25}$; $\frac{24}{7}$. (ii) $\frac{7}{25}$; $\frac{24}{25}$; $\frac{7}{24}$. **6.** (*a*) 45. (*b*) $\sqrt{2}$. (*c*) $\frac{1}{\sqrt{2}}$. **7.** (*b*) $\sqrt{3}$. (*c*) (i) $\frac{1}{2}$. (ii) $\frac{\sqrt{3}}{2}$.

Pages 386 and 387

1. (*a*) 6·43. (*b*) 16·4. (*c*) 2·07. **2.** (*a*) 42°, 18·6, 16·7. (*b*) 15°, 9·66, 2·59. (*c*) 36°, 2·91, 2·12. **3.** 14·8. **4.** 183. **5.** 10·7. **6.** (*a*) 7·07. (*b*) 4·24. (*c*) 0·619. **7.** 30·6. **8.** 4·70. **9.** (*a*) 0·108. (*b*) 0·0722. (*c*) 0·0217. **10.** 4·36. **11.** 115. **12.** (*a*) 6·43. (*b*) 7·66. (*c*) 7·34. (*d*) 9·8. (*e*) $41\frac{1}{2}°$. (*f*) $98\frac{1}{2}°$.

Pages 388 and 389

1. (*a*) 12°. (*b*) $14\frac{1}{2}°$. (*c*) $70\frac{1}{2}°$. **2.** (*a*) 13°, 77°. (*b*) 23°, 67°. (*c*) 16°, 74°. **3.** 122°. **4.** 4°. **5.** (*a*) 58°, 58°, 64°. (*b*) 18°, 81°, 81°. **6.** 62°. **7.** (*a*) 35°, 145°. (*b*) $39\frac{1}{2}°$, $50\frac{1}{2}°$. (*c*) 37°, 42°, 169°. (*d*) 29°.

Pages 389 and 390

1. 11·3. **2.** (*a*) 56·4. (*b*) 1·88. (*c*) 9·58. (*d*) 48·1. **3.** (*a*) 20·8. (*b*) 98·6. (*c*) 100. **4.** 72·5. **5.** (*a*) 67·5. (*b*) 89·0. (*c*) 59·4. (*d*) 181.

Pages 391 and 392

2. 14·6. **3.** 10·3. **4.** (*a*) 7·07. (*b*) 2·83. (*c*) 6·01. (*d*) 16·6. (*e*) 90·2. **5.** (*a*) 0·354. (*b*) 0·530. (*c*) 1·41. **6.** 6·6. **7.** 9·40. **8.** 9·18. **9.** 10·3. **10.** 10·1. **11.** 1·75. **12.** 5·83, 6·17. **13.** 6·93. **14.** 3·46. **15.** 8·19; 5·74.

Pages 393 and 394

1. (*a*) $45\frac{1}{2}°$. (*b*) 74°. (*c*) 39°. **2.** 74°. **3.** 29°. **4.** 71°, 71°, 38°. **5.** (*a*) 31°, 31°. (*b*) $83\frac{1}{2}°$, $83\frac{1}{2}°$, 13°. (*c*) 35°, 55°. (*d*) 53°, 37°. **6.** 58°. **7.** (*a*) 2,828. (*b*) 17,760. **8.** 30. **9.** 62°. **11.** 113° or 11°. **12.** 52°. **13.** (*a*) 37°. (*b*) 48°. (*c*) 95°.

Pages 396 and 397

1. 143. **2.** 405. **3.** 2,135. **4.** 10·4. **5.** 33·0. **6.** 106. **7.** 746. **8.** (*a*) 10·4. (*b*) 0·108. (*c*) 0·0217. **9.** 0·0650. **10.** (*a*) 10·4. (*b*) 0·866. **11.** 43·0. **12.** 6·63. **13.** 4·76. **14.** 276·2. **15.** 41·3. **16.** 1·44. **17.** 4·79.

Pages 398 and 399

1. 37°, 53°. **2.** (*a*) 32°, 58°. (*b*) $43\frac{1}{2}°$, $46\frac{1}{2}°$. (*c*) 13°, 77°. (*d*) 37°, 53°. **3.** 56°. **4.** 110°. **5.** 50°, 65°. **6.** 77°. **7.** 53°, 127°. **8.** 3°. **9.** $6\frac{1}{2}°$. **10.** $36\frac{1}{2}°$. **11.** 37°. **14.** (*a*) 5·6713. (*b*) 0·1051. **15.** (*a*) 8° 18′. (*b*) 30° 0′. (*c*) 1° 12′. (*d*) 1° 26′. (*e*) 1° 30′. (*f*) 0° 54′.

Pages 399, 400, 401, 402 and 403

Review Exercise

1. (*a*) 6·13, 5·14, 40°. (*b*) 12·9, 15·7, 35°. (*c*) $41\frac{1}{2}°$, $48\frac{1}{2}°$, 7·94. (*d*) 13·9, 30°, 60°. **2.** (*a*) 9°, 41·5, 6·57. (*b*) 50°, 6·94, 8·27. (*c*) 39°, 1·49, 1·21. **3.** (*a*) 40°, 17·9, 23·3. (*b*) 20°, 6·92, 20·2. (*c*) 32°, 1·44, 2·71. **4.** (*a*) 72·2. (*b*) 65·9. **5.** (*a*) 7, 74°, 16°. (*b*) 57°, 33°, 27·7. (*c*) 38°, 52°, 11·0. **6.** (*a*) 39, $67\frac{1}{4}°$, $22\frac{1}{2}°$. (*b*) 51, 62°, 28°. (*c*) 12·0, $48\frac{1}{2}°$, $41\frac{1}{2}°$. **7.** (*a*) 3·46. (*b*) 4·67. (*c*) 2·59. (*d*) 1·31. **8.** (*a*) 124. (*b*) 40·3. (*c*) 55·8. (*d*) 21·4. **9.** (*a*) 180. (*b*) 139. **10.** 9,540. **11.** (*a*) 10, 10. (*b*) 10, $27\frac{1}{2}$. (*c*) 10, $11\frac{15}{16}$. (*d*) 10, $21\frac{7}{16}$. (*e*) 10, 7. (*f*) 5, $28\frac{3}{8}$. **12.** 123·2. **13.** $32\frac{1}{2}$. **14.** (*a*) $19\frac{1}{2}$. (*b*) 80. **15.** (*a*) 10·9, 5·07. (*b*) 83·1, 17·7. **16.** (*a*) 2·73. (*b*) 3·75. **17.** (*a*) 3·06. (*b*) 2·57. (*c*) 2·43. (*d*) $51\frac{1}{2}$. (*e*) $78\frac{1}{2}$. (*f*) 7·66. **18.** (*a*) 75. (*b*) 9·66. (*c*) 2·59. (*d*) 11·5, 15·0. (*e*) 14·1. (*f*) 68·1. **19.** (*a*) 84. (*b*) 11·2. (*c*) $59\frac{1}{2}$. (*d*) 6·59. (*e*) 53. (*f*) 8·41. (*g*) $67\frac{1}{2}$. **20.** (*a*) 24·4. (*b*) 27·7. (*c*) 28·1. (*d*) 27·7. (*e*) 24·4. (*f*) 18·1. **21.** 4·70, 7·61, 7·61, 4·70.

NOTICE

This book is being lent to you subject to the following conditions:

1. You must handle it with care and not mark it in any way.

2. You must return it to the Correspondence School Branch by insured mail, postage prepaid, as soon as the course has been completed, discontinued, or cancelled.

3. Your name and file number must be written in the space provided in the book and also on the parcel.

Note:

If any book is badly soiled the student may be required to replace it. If any student has his own text or wishes to buy his own text, this one should be returned promptly to the Branch.

Student's name

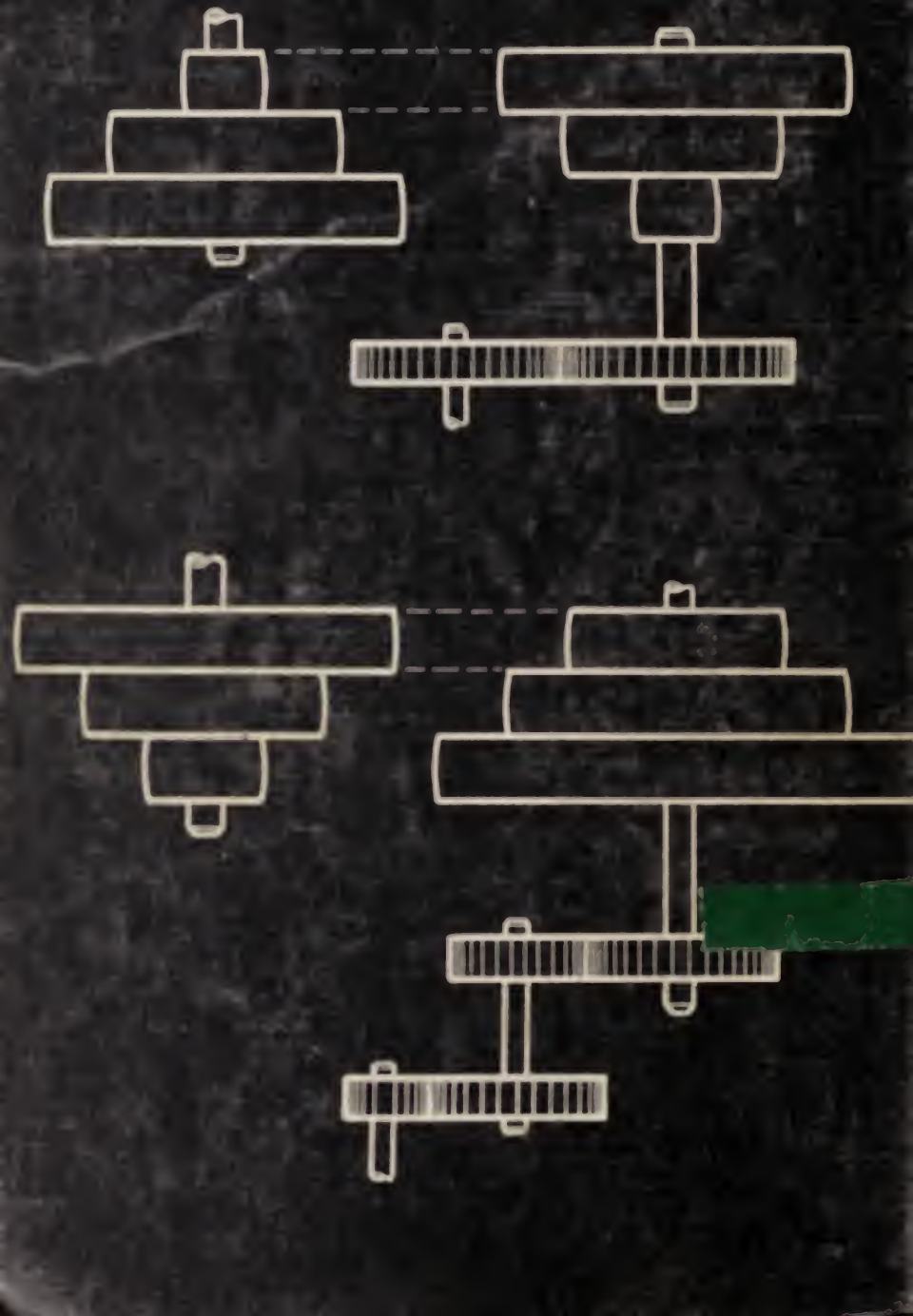
File number

Do not detach this slip

QA 39 B87 1964
BROWN ARTHUR ESPINER 1911-
MATHEMATICS IN PRACTICE/

39566248 CURR HIST

000011118205

QA 39 B87 1964
Brown, Arthur Espiner,1911-
Mathematics in practice /

39566248 CURR HIST